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CALCULUS 1 SURVIVAL PACK

130+

PRACTICE PROBLEMS & SOLUTIONS

LIMITS | DERIVATIVES | INTEGRALS | APPLICATIONS

- EXAM-FOCUSED
- STEP-BY-STEP SOLUTIONS
- METHOD RECOGNITION TIPS
- PERFECT FOR SELF-STUDY

HOW TO USE THIS SURVIVAL PACK

1 FOCUS ON PATTERN RECOGNITION

Calculus becomes MUCH easier once you recognize common problem types and solution patterns.

2 DON'T TRY TO MEMORIZE EVERYTHING

The goal is not to become a math genius overnight. Focus on understanding the most common exam-style questions first.

3 PRACTICE ACTIVELY

Don't just read the solutions. Attempt every problem yourself before checking the answer.

4 PRIORITIZE THE HIGH-IMPACT TOPICS

- ★ Review of functions
- ★ Limits
- ★ Derivatives
- ★ Integrals
- ★ Applications

> These topics appear most often on exams and carry the highest weight.

5 RECOMMENDED STUDY STRATEGY

1. Try the problem → 2. Get stuck → 3. Review the solution → 4. Identify the pattern → 5. Repeat similar questions

DIFFICULTY LEVELS

EASY	MEDIUM	HARD
Build your confidence	Stronger practice & understanding	More difficult questions

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SECTION 1

REVIEW OF FUNCTIONS

Strengthen your foundation by mastering the core concepts of functions.

INPUT & OUTPUT
Review of Functions

DIFFICULTY: Easy | Q #: 1.1

QUESTION

For the given function, find the following values:

$$f(x) = x^2 + 3x + 5$$

A $f(2)$

B $f(5)$

C $f(-3)$

TIP: Substitute the value for the variable and simplify!

SOLUTION

A $f(2)$

$$\begin{aligned} f(2) &= (2)^2 + 3(2) + 5 \\ &= 4 + 6 + 5 \\ &= \boxed{15} \end{aligned}$$

B $f(5)$

$$\begin{aligned} f(5) &= (5)^2 + 3(5) + 5 \\ &= 25 + 15 + 5 \\ &= \boxed{45} \end{aligned}$$

C $f(-3)$

$$\begin{aligned} f(-3) &= (-3)^2 + 3(-3) + 5 \\ &= 9 - 9 + 5 \\ &= \boxed{5} \end{aligned}$$

FINAL ANSWER:

$$A.\ 15,\quad B.\ 45,\quad C.\ 5$$

FUNCTION OR NOT

Review of Functions

DIFFICULTY: Medium | Q #: 1.2

QUESTION

Determine whether the relation represents y as a function of x. Justify your answer.

A. $\{(-2,4),\ (-1,0),\ (1,3),\ (2,-1),\ (3,5)\}$

B. $\{(-2,4),\ (-1,2),\ (-1,-3),\ (0,1),\ (2,2)\}$

C. $y = x^2 - 2x + 1$

D. $x^2 + y^2 = 9$

TIP:

A relation represents y as a function of x if every x-value is paired with exactly one y-value.

Use the vertical line test for graphs or check repeated x-values in lists of ordered pairs.

SOLUTION

A Function

All x-values are different, so each x is paired with exactly one y-value.

Answer: Function

B Not a function

The x -value -1 is paired with two different y-values (2 and -3).

Answer: Not a function

C Function

This is a quadratic equation. For every x, there is exactly one y.

Answer: Function

D Not a function

This is a circle. For many x-values (e.g., $x = 0$), there are two y-values ($y = 3$ and $y = -3$).

Answer: Not a function

FINAL ANSWER:

$A.$ Function, $B.$ Not a function, $C.$ Function, $D.$ Not a function

DOMAIN AND RANGE

Review of Functions

DIFFICULTY: Easy | Q #: 1.3

QUESTION

For each function, determine the domain and range. Express your answers in interval notation.

A $f(x) = 3x - 2$

B $g(x) = \dfrac{2}{x-1}$

C $h(x) = \sqrt{x+4}$

TIP:

- Linear functions: domain and range are all real numbers.
- Rational functions: exclude values that make the denominator zero.
- Square root functions: require the expression inside the root to be ≥ 0.

SOLUTION

A $f(x) = 3x - 2$

- Linear function → no restrictions on x or y.
- Domain and range are all real numbers.

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

B $g(x) = \dfrac{2}{x-1}$

- Denominator cannot be zero: $x - 1 \neq 0 \Rightarrow x \neq 1.$
- The function can never equal 0, so $y \neq 0.$

Domain: $(-\infty, 1) \cup (1, \infty)$

Range: $(-\infty, 0) \cup (0, \infty)$

C $h(x) = \sqrt{x+4}$

- Expression inside the square root must be ≥ 0:
  $x + 4 \geq 0 \Rightarrow x \geq -4.$
- Square root outputs are always $\geq 0.$

Domain: $[-4, \infty)$

Range: $[0, \infty)$

FINAL ANSWER:

$A.$ Domain: $(-\infty, \infty)$, *Range*: $(-\infty, \infty)$

$B.$ Domain: $(-\infty, 1) \cup (1, \infty)$, *Range*: $(-\infty, 0) \cup (0, \infty)$

$C.$ Domain: $[-4, \infty)$, *Range*: $[0, \infty)$

DOMAIN AND RANGE
Review of Functions

DIFFICULTY: Medium | Q #: 1.4

QUESTION

For each function, determine the domain and range.

Express your answers in interval notation.

A. $f(x) = \dfrac{x^2 + 3x - 4}{x - 1}$

B. $g(x) = \sqrt{x^2 - 5x + 6}$

SOLUTION

A. $f(x) = \dfrac{x^2 + 3x - 4}{x - 1} = \dfrac{(x+4)(x-1)}{x-1}$

$$= x + 4, \quad x \neq 1$$

- Denominator cannot be zero: $x - 1 \neq 0 \Rightarrow x \neq 1$.

Domain: $(-\infty, 1) \cup (1, \infty)$

Range: $(-\infty, 5) \cup (5, \infty)$

B. $g(x) = \sqrt{x^2 - 5x + 6}$

- Require $x^2 - 5x + 6 \geq 0$.
- Factor: $(x-2)(x-3) \geq 0$.
- Solution: $x \leq 2$ or $x \geq 3$.
- Square root outputs are always ≥ 0.

Domain: $(-\infty, 2] \cup [3, \infty)$

Range: $[0, \infty)$

FINAL ANSWER:

A. Domain: $(-\infty, 1) \cup (1, \infty)$, Range: $(-\infty, 5) \cup (5, \infty)$

B. Domain: $(-\infty, 2] \cup [3, \infty)$, Range: $[0, \infty)$

QUESTION

Given the functions

$$f(x) = 4x - 3 \quad \text{and} \quad g(x) = x^2 - 2$$

find the following.

A. $(f \circ g)(x)$

B. $(g \circ f)(x)$

SOLUTION

A. $(f \circ g)(x)$

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= f(x^2 - 2) \\
&= 4(x^2 - 2) - 3 \\
&= 4x^2 - 8 - 3 \\
&= \boxed{\mathbf{4x^2 - 11}}
\end{aligned}$$

B. $(g \circ f)(x)$

$$\begin{aligned}
(g \circ f)(x) &= g(f(x)) \\
&= g(4x - 3) \\
&= (4x - 3)^2 - 2 \\
&= 16x^2 - 24x + 9 - 2 \\
&= 16x^2 - 24x + 7 \\
&= \boxed{\mathbf{16x^2 - 24x + 7}}
\end{aligned}$$

QUESTION

Given the functions

$$f(x) = \frac{2x+3}{x-1}$$

$$g(x) = \sqrt{x+5}$$

Find:

A. $(f \circ g)(x)$

Determine the domain of the composite function.

SOLUTION

A. $(f \circ g)(x)$

$$\begin{aligned} f(g(x)) &= f(\sqrt{x+5}) \\ &= \frac{2\sqrt{x+5}+3}{\sqrt{x+5}-1} \end{aligned}$$

Domain restrictions:

1. From the square root:

$$x + 5 \geq 0$$
$$x \geq -5$$

2. From the denominator:

$$\sqrt{x+5} - 1 \neq 0$$
$$\sqrt{x+5} \neq 1$$
$$x + 5 \neq 1$$
$$x \neq -4$$

Composite Function:

$$(f \circ g)(x) = \frac{2\sqrt{x+5}+3}{\sqrt{x+5}-1}$$

Domain: $[-5, -4) \cup (-4, \infty)$

FINAL ANSWER:

A. $(f \circ g)(x) = \frac{2\sqrt{x+5}+3}{\sqrt{x+5}-1}$

Domain: $[-5, -4) \cup (-4, \infty)$

VERTICAL ASYMPTOTES
Rational Functions

DIFFICULTY: Easy | Q #: **1.7**

QUESTIONS

Find all vertical asymptotes of each rational function.

A $f(x) = \dfrac{2x^2 - 3x + 1}{x^2 - 4x}$

B $g(x) = \dfrac{x^2 + 2x - 3}{x^2 + x - 6}$

TIP

Vertical asymptotes occur where the denominator is zero and the numerator is nonzero.

A Factor the denominator completely.

B Set each factor equal to zero and check that the numerator is not zero at those values.

SOLUTIONS

A $f(x) = \dfrac{2x^2 - 3x + 1}{x^2 - 4x}$

1. **Factor the denominator:**

$$x^2 - 4x = x(x - 4)$$

2. **Set each factor equal to zero:**

$$x = 0 \quad \text{or} \quad x - 4 = 0$$
$$x = 0 \quad \text{or} \quad x = 4$$

3. **Check that the numerator is nonzero at these values.**

$$f(0) = 2(0)^2 - 3(0) + 1 = 1 \neq 0$$
$$f(4) = 2(4)^2 - 3(4) + 1 = 25 \neq 0$$

FINAL ANSWER:

$$x = 0 \text{ and } x = 4$$

B $g(x) = \dfrac{x^2 + 2x - 3}{x^2 + x - 6}$

1. **Factor the denominator:**

$$x^2 + x - 6 = (x + 3)(x - 2)$$

2. **Set each factor equal to zero:**

$$x + 3 = 0 \quad \text{or} \quad x - 2 = 0$$
$$x = -3 \quad \text{or} \quad x = 2$$

3. **Check that the numerator is nonzero at these values.**

$$g(-3) = (-3)^2 + 2(-3) - 3 = 9 - 6 - 3 = 0$$
$$g(2) = (2)^2 + 2(2) - 3 = 4 + 4 - 3 = 5 \neq 0$$

FINAL ANSWER:

$$x = 2$$

HORIZONTAL ASYMPTOTES

Rational Functions

DIFFICULTY: Medium

Q #: 1.8

QUESTIONS

Find the horizontal asymptote of each rational function.

A $f(x) = \dfrac{3x^2 + 2x - 1}{x^2 - 4x + 5}$

B $g(x) = \dfrac{5x^2 - 7x + 8}{2x - 3}$

C $h(x) = \dfrac{2x + 1}{4x^2 - x + 6}$

TIP

To find the horizontal asymptote of $f(x) = \frac{p(x)}{q(x)}$:

- If $\deg\, p(x) = \deg\, q(x)$: Horizontal asymptote is $y = \dfrac{\text{leading coefficient of } p}{\text{leading coefficient of } q}$
- If $\deg\, p(x) < \deg\, q(x)$: Horizontal asymptote is $y = 0$
- If $\deg\, p(x) > \deg\, q(x)$: There is no horizontal asymptote.

SOLUTIONS

A $f(x) = \dfrac{3x^2 + 2x - 1}{x^2 - 4x + 5}$

- $\deg(3x^2 + 2x - 1) = 2$ and $\deg(x^2 - 4x + 5) = 2$ (Degrees are equal.)
- Horizontal asymptote is the ratio of the leading coefficients: $y = \dfrac{3}{1} = 3$

FINAL ANSWER:

$$y = 3$$

B $g(x) = \dfrac{5x^2 - 7x + 8}{2x - 3}$

- $\deg(5x^2 - 7x + 8) = 2$ and $\deg(2x - 3) = 1$ (Degree of numerator is greater.)
- Since $\deg\, p(x) > \deg\, q(x)$, there is no horizontal asymptote.

FINAL ANSWER:

No horizontal asymptote

C $h(x) = \dfrac{2x + 1}{4x^2 - x + 6}$

- $\deg(2x + 1) = 1$ and $\deg(4x^2 - x + 6) = 2$ (Degree of numerator is less.)
- Horizontal asymptote is $y = 0$.

FINAL ANSWER:

$$y = 0$$

OBLIQUE ASYMPTOTES
Rational Functions

DIFFICULTY: Medium | Q #: 1.9

QUESTIONS

Find the oblique asymptote of the rational function. Show all work.

$$f(x) = \frac{3x^2 - x + 2}{x - 2}$$

TIP

If the degree of the numerator is exactly one more than the degree of the denominator, the function has an oblique (slant) asymptote.

To find it:

1. Perform polynomial long division: divide the numerator by the denominator.
2. The quotient (without remainder) is the equation of the oblique asymptote.

SOLUTION

Since the degree of the numerator (2) is one more than the degree of the denominator (1), the function has an oblique asymptote.

1. Perform polynomial long division:

$$\begin{array}{r} 3x + 5 \\ x - 2 \overline{\smash{)}\, 3x^2 - x + 2} \\ -(3x^2 - 6x) \\ \hline 5x + 2 \\ -(5x - 10) \\ \hline 12 \end{array}$$

2. The quotient is $3x + 5$.
3. Therefore, the oblique asymptote is:

FINAL ANSWER:

$$y = 3x + 5$$

SOLVING TRIG EQUATIONS
Trigonometry

DIFFICULTY: Easy | Q #: 1.10

QUESTION

Solve for x in the interval $0° \leq x < 360°$.

$$\sin x = \frac{\sqrt{3}}{2}$$

TIP

To solve $\sin x = k$ in $0° \leq x < 360°$:

1. Find the reference angle θ such that $\sin\theta = k$.
2. Since sine is positive in Quadrants I and II, the solutions are $x = \theta$ and $x = 180° - \theta$.

SOLUTION

The reference angle for $\sin x = \dfrac{\sqrt{3}}{2}$ is $60°$ since $\sin 60° = \dfrac{\sqrt{3}}{2}$.

Sine is positive in Quadrants I and II.

1. **Quadrant I:**

$$x = 60°$$

2. **Quadrant II:**

$$x = 180° - 60° = 120°$$

Therefore, the solutions in $0° \leq x < 360°$ are:

FINAL ANSWER:

$$x = 60°,\ 120°$$

SOLVING TRIG EQUATIONS

Trigonometry

DIFFICULTY: Medium

Q #: 1.11

QUESTION

Solve for x in the interval $0° \leq x < 360°$.

$$2\cos x - 1 = 0$$

SOLUTION

1. Isolate cosine:

$$2\cos x - 1 = 0$$

$$2\cos x = 1$$

$$\cos x = \frac{1}{2}$$

2. Find the reference angle:

$$\cos\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = 60°$$

3. Determine the solutions in $0° \leq x < 360°$.

Cosine is positive in Quadrants I and IV.

- **Quadrant I:** $x = \theta = 60°$
- **Quadrant IV:** $x = 360° - \theta = 360° - 60° = 300°$

FINAL ANSWER:

$$x = 60°, 300°$$

TIP

To solve $a\cos x + b = 0$:

1. Isolate cosine: $\cos x = -\dfrac{b}{a}$.
2. Find the reference angle θ such that $\cos\theta = \left|\dfrac{b}{a}\right|$.
3. Since cosine is positive in Quadrants I and IV, the solutions are $x = \theta$ and $x = 360° - \theta$.

GRAPHING EXPONENTIAL FUNCTIONS
Exponential Functions

DIFFICULTY: Easy | Q #: **1.12**

QUESTION

Graph the exponential function.

Show all work.

$$y = 3e^x$$

TIP

To graph an exponential function $y = ae^x$:

1. Choose several x-values. Evaluate $y = ae^x$.
2. Plot the points and draw a smooth increasing curve.

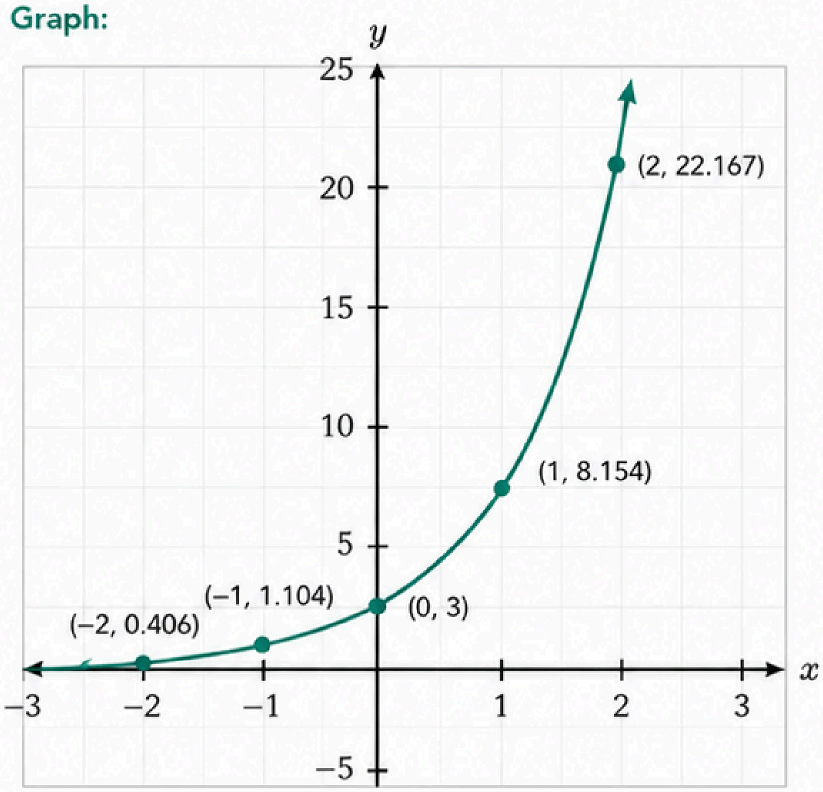
SOLUTION

Choose five values for x and evaluate $y = 3e^x$.

T-chart:

x	$y = 3e^x$	(x, y)
-2	$3e^{-2} = \frac{3}{e^2} \approx 0.406$	$(-2, 0.406)$
-1	$3e^{-1} = \frac{3}{e} \approx 1.104$	$(-1, 1.104)$
0	$3e^0 = 3$	$(0, 3)$
1	$3e^1 = 3e \approx 8.154$	$(1, 8.154)$
2	$3e^2 = 3e^2 \approx 22.167$	$(2, 22.167)$

Graph:



SIMPLE LOGARITHMS
Logarithmic Functions

DIFFICULTY: Easy | Q #: 1.13

QUESTION

Solve for x in each equation.

A $\log_2 x = 3$

B $\log_5 (x-2) = 2$

C $\log_3 (2x+1) = 4$

TIP

To solve $\log_b A = k$ (where $b > 0$, $b \neq 1$):

Rewrite in exponential form: $A = b^k$ and solve for x.

SOLUTION

A $\log_2 x = 3$

- Rewrite in exponential form: $x = 2^3$
- $x = 8$

FINAL ANSWER:

$$x = 8$$

B $\log_5 (x-2) = 2$

- Rewrite in exponential form: $x - 2 = 5^2$
- $x - 2 = 25$
- $x = 27$

FINAL ANSWER:

$$x = 27$$

C $\log_3 (2x+1) = 4$

- Rewrite in exponential form: $2x + 1 = 3^4$
- $2x + 1 = 81$
- $2x = 80$
- $x = 40$

FINAL ANSWER:

$$x = 40$$

SIMPLE LOGARITHMS

Logarithmic Functions

DIFFICULTY: Medium | Q #: 1.14

QUESTION

Solve for x in each equation.

A $\log_3(2x - 1) = 5$

B $\log_4(x^2 - 2x + 1) = 3$

C $\log_5(3x + 4) = \log_5(x^2 - 7)$

TIP

To solve $\log_b A = k$ (where $b > 0$, $b \neq 1$):

Rewrite in exponential form: $A = b^k$ and solve for x.

SOLUTION

A $\log_3(2x - 1) = 5$

- Rewrite in exponential form: $2x - 1 = 3^5$
- $2x - 1 = 243$
- $2x = 244$
- $x = 122$

FINAL ANSWER:

$$x = 122$$

B $\log_4(x^2 - 2x + 1) = 3$

- Rewrite in exponential form: $x^2 - 2x + 1 = 4^3$
- $x^2 - 2x + 1 = 64$
- $(x - 1)^2 = 64$
- $x - 1 = \pm 8$
- $x = 9$ or $x = -7$

FINAL ANSWER:

$$x = 9 \text{ or } x = -7$$

C $\log_5(3x + 4) = \log_5(x^2 - 7)$

- Since the bases are the same, set the arguments equal:
$$3x + 4 = x^2 - 7$$
- $x^2 - 3x - 11 = 0$
- $x = \dfrac{3 \pm \sqrt{53}}{2}$

FINAL ANSWER:

$$x = \frac{3 \pm \sqrt{53}}{2}$$

SOLVING LOGARITHMIC EQUATIONS

Logarithmic equations

DIFFICULTY: Medium

Q #: 1.15

QUESTION

Solve for x in each equation.

A $\log_2(x) + \log_2(x-3) = 3$

B $\log_4(x+2) - \log_4(x-1) = 1$

C $2\log_5(x+1) + \log_5 3 = 4$

TIP

Useful logarithmic properties:

$$\log_a(xy) = \log_a(x) + \log_a(y)$$

$$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$$

$$\log_a(x^r) = r\log_a(x)$$

SOLUTION

A $\log_2(x) + \log_2(x-3) = 3$

- $\log_2[x(x-3)] = 3$
- $x(x-3) = 2^3$
- $x^2 - 3x - 8 = 0$
- $x = \dfrac{3 \pm \sqrt{41}}{2}$

FINAL ANSWER:

$$x = \frac{3+\sqrt{41}}{2}$$

B $\log_4(x+2) - \log_4(x-1) = 1$

- $\log_4\left(\dfrac{x+2}{x-1}\right) = 1$
- $\dfrac{x+2}{x-1} = 4^1 = 4$
- $x + 2 = 4(x-1)$
- $x + 2 = 4x - 4$
- $6 = 3x$
- $x = 2$

FINAL ANSWER: $x = 2$

C $2\log_5(x+1) + \log_5 3 = 4$

- $\log_5[(x+1)^2] + \log_5 3 = 4$
- $\log_5[3(x+1)^2] = 4$
- $3(x+1)^2 = 5^4$
- $(x+1)^2 = \dfrac{625}{3}$
- $x = -1 \pm \dfrac{25\sqrt{3}}{3}$

FINAL ANSWER:

$$x = -1 + \frac{25\sqrt{3}}{3} \quad \text{or} \quad x = -1 - \frac{25\sqrt{3}}{3}$$

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SECTION 2

LIMITS

Strengthen your foundation by mastering the core concepts of limits.

FINDING LIMITS USING A TABLE
Limits

DIFFICULTY: Easy | Q #: 2.1

QUESTION

Use the table of values for the function $f(x)$ to find $\lim_{x \to 2} f(x)$.

x	$f(x)$
1.5	3.1
1.8	3.7
1.9	3.9
1.99	3.99
2.01	4.01
2.1	4.1
2.2	4.3
2.5	4.7

What is the value of $\lim_{x \to 2} f(x)$?

SOLUTION

To find $\lim_{x \to 2} f(x)$, look at the values of $f(x)$ as x gets closer to 2 from both the left and the right.

From the left ($x < 2$):

As x gets closer to 2, $f(x)$ gets closer to 4.

$(3.1,\ 3.7,\ 3.9,\ 3.99\ \to 4)$

From the right ($x > 2$):

As x gets closer to 2, $f(x)$ gets closer to 4.

$(4.01,\ 4.1,\ 4.3,\ 4.7\ \to 4)$

Since the values approach 4 from both sides,

$\lim_{x \to 2} f(x) = 4.$

FINAL ANSWER:

$$\lim_{x \to 2} f(x) = 4$$

FINDING LIMITS FROM GRAPH

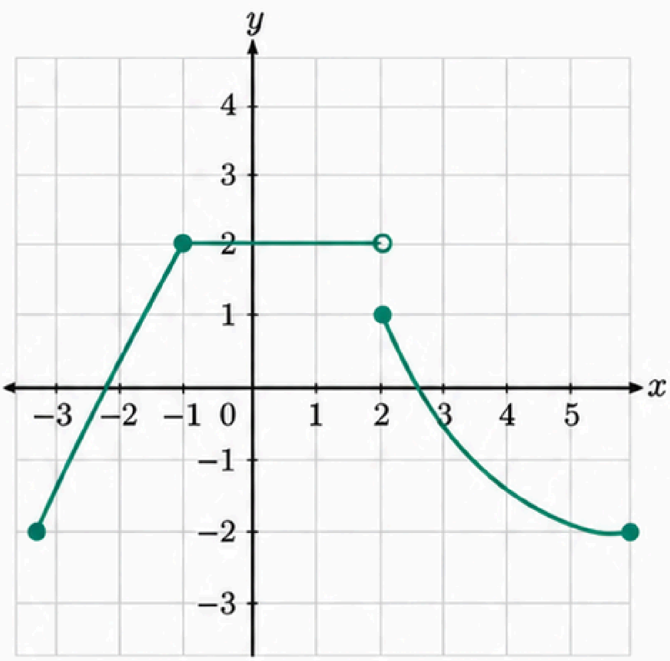
ONE SIDED LIMITS

DIFFICULTY: Medium

Q #: 2.2

QUESTION

Use the graph of $f(x)$ to find the following limits.



a) $\lim_{x \to 2^-} f(x)$

b) $\lim_{x \to 2^+} f(x)$

c) $\lim_{x \to 2} f(x)$

d) $\lim_{x \to -1^-} f(x)$

e) $\lim_{x \to -1^+} f(x)$

f) $\lim_{x \to -1} f(x)$

SOLUTION

a) $\lim_{x \to 2^-} f(x)$

As x approaches 2 from the left, $f(x)$ approaches 2.

$$\boxed{\lim_{x \to 2^-} f(x) = 2}$$

b) $\lim_{x \to 2^+} f(x)$

As x approaches 2 from the right, $f(x)$ approaches 1.

$$\boxed{\lim_{x \to 2^+} f(x) = 1}$$

c) $\lim_{x \to 2} f(x)$

The left-hand limit is 2 and the right-hand limit is 1. Since they are not equal, the two-sided limit does not exist.

$$\boxed{\lim_{x \to 2} f(x) \text{ does not exist}}$$

d) $\lim_{x \to -1^-} f(x)$

As x approaches -1 from the left, $f(x)$ approaches 2.

$$\boxed{\lim_{x \to -1^-} f(x) = 2}$$

e) $\lim_{x \to -1^+} f(x)$

As x approaches -1 from the right, $f(x)$ approaches 2.

$$\boxed{\lim_{x \to -1^+} f(x) = 2}$$

f) $\lim_{x \to -1} f(x)$

Both one-sided limits are 2, so the two-sided limit exists and equals 2.

$$\boxed{\lim_{x \to -1} f(x) = 2}$$

QUESTION

Evaluate the following limits.

a) $\lim_{x \to 3} (2x + 5)$

b) $\lim_{x \to -2} (x^2 - 4x + 1)$

c) $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$

SOLUTION

a) $\lim_{x \to 3} (2x + 5)$

Direct substitution:

$= 2(3) + 5$

$= 6 + 5$

$\boxed{= 11}$

b) $\lim_{x \to -2} (x^2 - 4x + 1)$

Direct substitution:

$= (-2)^2 - 4(-2) + 1$

$= 4 + 8 + 1$

$\boxed{= 13}$

c) $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$

Direct substitution gives an indeterminate form:

$\frac{1^2 - 1}{1 - 1} \qquad \frac{0}{0}$

Factor the numerator:

$= \lim_{x \to 1} \frac{\cancel{(x - 1)}(x + 1)}{\cancel{x - 1}}$

$= \lim_{x \to 1} (x + 1)$

$= 1 + 1$

$\boxed{= 2}$

QUESTION

Evaluate the following limits.

a) $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$

b) $\lim_{x \to 1} \frac{\sqrt{x+3} - 2}{x - 1}$

c) $\lim_{x \to -2} \frac{x^2 + 5x + 6}{x^2 + 3x + 2}$

SOLUTION

a) $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$

Factor the numerator:

$$= \lim_{x \to 3} \frac{\cancel{(x-3)}(x+3)}{\cancel{x-3}}$$

$$= \lim_{x \to 3} (x+3)$$

$$= 3 + 3$$

FINAL ANSWER

$\boxed{6}$

b) $\lim_{x \to 1} \frac{\sqrt{x+3} - 2}{x - 1}$

Rationalize the numerator:

$$= \lim_{x \to 1} \frac{\sqrt{x+3} - 2}{x - 1} \cdot \frac{\sqrt{x+3} + 2}{\sqrt{x+3} + 2}$$

$$= \lim_{x \to 1} \frac{x + 3 - 4}{(x-1)(\sqrt{x+3} + 2)}$$

$$= \lim_{x \to 1} \frac{\cancel{x-1}}{\cancel{(x-1)}(\sqrt{x+3} + 2)}$$

$$= \lim_{x \to 1} \frac{1}{\sqrt{x+3} + 2}$$

$$= \frac{1}{\sqrt{1+3} + 2}$$

$$= \frac{1}{2+2}$$

FINAL ANSWER $\boxed{\frac{1}{4}}$

c) $\lim_{x \to -2} \frac{x^2 + 5x + 6}{x^2 + 3x + 2}$

Factor numerator and denominator:

$$= \lim_{x \to -2} \frac{\cancel{(x+2)}(x+3)}{\cancel{(x+2)}(x+1)}$$

$$= \lim_{x \to -2} \frac{x+3}{x+1}$$

$$= \frac{-2+3}{-2+1}$$

$$= \frac{1}{-1}$$

FINAL ANSWER $\boxed{-1}$

QUESTION

Evaluate the following limit.

a) $\lim_{x\to 0} \dfrac{\sqrt{e^x+1}-\sqrt{2}}{e^x-1}$

TIP

If you have a difference of square roots in the numerator, multiply by the conjugate:

$$\sqrt{A}-\sqrt{B} \quad \times \quad \frac{\sqrt{A}+\sqrt{B}}{\sqrt{A}+\sqrt{B}}$$

This removes the radicals in the numerator.

SOLUTION

Direct substitution gives an indeterminate form:

$$\lim_{x\to 0} \frac{\sqrt{e^x+1}-\sqrt{2}}{e^x-1} = \frac{\sqrt{e^0+1}-\sqrt{2}}{e^0-1} = \frac{\sqrt{2}-\sqrt{2}}{1-1} = \frac{0}{0},$$

so rationalize.

Rationalize the numerator by multiplying by the conjugate $\sqrt{e^x+1}+\sqrt{2}$:

$$\lim_{x\to 0} \frac{\sqrt{e^x+1}-\sqrt{2}}{e^x-1} \cdot \frac{\sqrt{e^x+1}+\sqrt{2}}{\sqrt{e^x+1}+\sqrt{2}}$$

$$= \lim_{x\to 0} \frac{\left(\sqrt{e^x+1}\right)^2-\left(\sqrt{2}\right)^2}{(e^x-1)\left(\sqrt{e^x+1}+\sqrt{2}\right)}$$

$$= \lim_{x\to 0} \frac{e^x+1-2}{(e^x-1)\left(\sqrt{e^x+1}+\sqrt{2}\right)}$$

$$= \lim_{x\to 0} \frac{e^x-1}{(e^x-1)\left(\sqrt{e^x+1}+\sqrt{2}\right)}$$

Cancel the common factor e^x-1:

$$= \lim_{x\to 0} \frac{1}{\sqrt{e^x+1}+\sqrt{2}}$$

Now substitute $x=0$:

$$= \frac{1}{\sqrt{e^0+1}+\sqrt{2}}$$
$$= \frac{1}{\sqrt{1+1}+\sqrt{2}}$$
$$= \frac{1}{\sqrt{2}+\sqrt{2}}$$
$$= \frac{1}{2\sqrt{2}}$$

FINAL ANSWER

$$\boxed{\frac{1}{2\sqrt{2}}}$$

DIFFICULTY: Medium | Q #: 2.6

QUESTION

Evaluate the following limits.

a) $$\lim_{x \to 2^-} \frac{1}{(2-x)^2}$$

b) $$\lim_{x \to 0^-} \frac{1}{x}$$

c) $$\lim_{x \to -3^-} \frac{-5}{x+3}$$

TIP

If a denominator approaches 0 and the numerator approaches a nonzero number:

- From the positive side $\rightarrow +\infty$
- From the negative side $\rightarrow -\infty$

SOLUTION

a) $$\lim_{x \to 2^-} \frac{1}{(2-x)^2}$$

As $x \to 2^-$, $(2-x) \to 0^+$.

Since $(2-x)^2 \to 0^+$ and the numerator is 1 (> 0),

the limit is $+\infty$.

$$\lim_{x \to 2^-} \frac{1}{(2-x)^2} = +\infty$$

b) $$\lim_{x \to 0^-} \frac{1}{x}$$

As $x \to 0^-$, $x \to 0^-$.

Since the numerator is 1 (> 0),

the limit is $-\infty$.

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

c) $$\lim_{x \to -3^-} \frac{-5}{x+3}$$

As $x \to -3^-$, $(x+3) \to 0^-$.

The numerator is -5 (< 0),

so (negative)/(negative small) $\rightarrow +\infty$.

$$\lim_{x \to -3^-} \frac{-5}{x+3} = +\infty$$

FINAL ANSWERS

a) $$\lim_{x \to 2^-} \frac{1}{(2-x)^2} = +\infty$$

b) $$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

c) $$\lim_{x \to -3^-} \frac{-5}{x+3} = +\infty$$

INFINITE LIMITS

Evaluating Limits

DIFFICULTY: Medium | Q #: 2.7

QUESTION

Consider the function

$$f(x) = \frac{3}{(x-2)^2}$$

Evaluate the following limits.

a) $\lim_{x\to 2^+} f(x)$

b) $\lim_{x\to 2^-} f(x)$

c) $\lim_{x\to 2} f(x)$

SOLUTION

a) $\lim_{x\to 2^+} \frac{3}{(x-2)^2}$

As $x \to 2^+$, $(x-2)^2 \to 0^+$.
Since the numerator is 3 (> 0),
the limit is $+\infty$.

$$\lim_{x\to 2^+} \frac{3}{(x-2)^2} = +\infty$$

b) $\lim_{x\to 2^-} \frac{3}{(x-2)^2}$

As $x \to 2^-$, $(x-2)^2 \to 0^+$.
Since the numerator is 3 (> 0),
the limit is $+\infty$.

$$\lim_{x\to 2^-} \frac{3}{(x-2)^2} = +\infty$$

c) $\lim_{x\to 2} \frac{3}{(x-2)^2}$

Both one-sided limits are $+\infty$,
so the two-sided limit is $+\infty$.

$$\lim_{x\to 2} \frac{3}{(x-2)^2} = +\infty$$

FINAL ANSWERS

a) $\lim_{x\to 2^+} f(x) = +\infty$

b) $\lim_{x\to 2^-} f(x) = +\infty$

c) $\lim_{x\to 2} f(x) = +\infty$

DIFFICULTY: Easy

Q #: 2.8

QUESTION

Evaluate the following limit.

$$\lim_{x\to\infty} \frac{7x^2 - 6x + 4}{2x^2 + x - 3}$$

TIP

For limits of rational functions as $x \to \infty$:

- If the degrees are equal, the limit is the ratio of the leading coefficients.

SOLUTION

Factor out the highest degree term from the numerator and denominator.

$$\lim_{x\to\infty} \frac{7x^2 - 6x + 4}{2x^2 + x - 3} = \lim_{x\to\infty} \frac{x^2\left(7 - \frac{6}{x} + \frac{4}{x^2}\right)}{x^2\left(2 + \frac{1}{x} - \frac{3}{x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{7 - \frac{6}{x} + \frac{4}{x^2}}{2 + \frac{1}{x} - \frac{3}{x^2}}$$

As $x \to \infty$, $\frac{6}{x} \to 0$, $\frac{4}{x^2} \to 0$, $\frac{1}{x} \to 0$, $\frac{3}{x^2} \to 0$.

$$= \lim_{x\to\infty} \frac{7 + 0 + 0}{2 + 0 - 0}$$

$$= \frac{7}{2}$$

$$\lim_{x\to\infty} \frac{7x^2 - 6x + 4}{2x^2 + x - 3} = \frac{7}{2}$$

FINAL ANSWER

$$\lim_{x\to\infty} \frac{7x^2 - 6x + 4}{2x^2 + x - 3} = \frac{7}{2}$$

QUESTION

Evaluate the following limit.

$$\lim_{x \to \infty} \frac{-4x^2 + 3x - 1}{2x^2 + 5}$$

TIP

For limits of rational functions as $x \to \infty$:

- If the degrees are equal, the limit is the ratio of the leading coefficients.

SOLUTION

Factor out the highest degree term from the numerator and denominator.

$$\lim_{x \to \infty} \frac{-4x^2 + 3x - 1}{2x^2 + 5} = \lim_{x \to \infty} \frac{x^2\left(-4 + \frac{3}{x} - \frac{1}{x^2}\right)}{x^2\left(2 + \frac{5}{x^2}\right)}$$

$$= \lim_{x \to \infty} \frac{-4 + \frac{3}{x} - \frac{1}{x^2}}{2 + \frac{5}{x^2}}$$

As $x \to \infty, \ \frac{3}{x} \to 0, \ \frac{1}{x^2} \to 0, \ \frac{5}{x^2} \to 0.$

$$= \lim_{x \to \infty} \frac{-4 + 0 - 0}{2 + 0}$$

$$= -\frac{4}{2}$$

$$= -2$$

$$\lim_{x \to \infty} \frac{-4x^2 + 3x - 1}{2x^2 + 5} = -2$$

FINAL ANSWER

$$\lim_{x \to \infty} \frac{-4x^2 + 3x - 1}{2x^2 + 5} = -2$$

QUESTION

Evaluate the following limit.

$$\lim_{x\to\infty} \frac{4x^3 - 7x^2 + 6x - 8}{2x^3 - 5x + 3}$$

TIP

For limits of rational functions as $x \to \infty$:

- If the degrees are equal, the limit is the ratio of the leading coefficients.

SOLUTION

Both the numerator and denominator have degree 3, so use the ratio of the leading coefficients.

$$\lim_{x\to\infty} \frac{4x^3 - 7x^2 + 6x - 8}{2x^3 - 5x + 3}$$

$$= \lim_{x\to\infty} \frac{x^3\left(4 - \frac{7}{x} + \frac{6}{x^2} - \frac{8}{x^3}\right)}{x^3\left(2 - \frac{5}{x^2} + \frac{3}{x^3}\right)}$$

$$= \lim_{x\to\infty} \frac{4 - \frac{7}{x} + \frac{6}{x^2} - \frac{8}{x^3}}{2 - \frac{5}{x^2} + \frac{3}{x^3}}$$

As $x \to \infty$, $\frac{7}{x} \to 0$, $\frac{6}{x^2} \to 0$, $\frac{8}{x^3} \to 0$, $\frac{5}{x^2} \to 0$, $\frac{3}{x^3} \to 0$.

$$= \lim_{x\to\infty} \frac{4 + 0 + 0 - 0}{2 - 0 + 0}$$

$$= \frac{4}{2}$$

$$= 2$$

$$\lim_{x\to\infty} \frac{4x^3 - 7x^2 + 6x - 8}{2x^3 - 5x + 3} = 2$$

FINAL ANSWER

$$\lim_{x\to\infty} \frac{4x^3 - 7x^2 + 6x - 8}{2x^3 - 5x + 3} = 2$$

QUESTION

Evaluate the following limit.

$$\lim_{x \to -\infty} \frac{8 + x - 4x^2}{\sqrt{6 + x^2 + 7x^4}}$$

SOLUTION

$$\lim_{x \to -\infty} \frac{8 + x - 4x^2}{\sqrt{6 + x^2 + 7x^4}} =$$

$$= \lim_{x \to -\infty} \frac{x^2\left(\frac{8}{x^2} + \frac{1}{x} - 4\right)}{\sqrt{x^4\left(\frac{6}{x^4} + \frac{1}{x^2} + 7\right)}}$$

$$= \lim_{x \to -\infty} \frac{x^2\left(\frac{8}{x^2} + \frac{1}{x} - 4\right)}{x^2\sqrt{\frac{6}{x^4} + \frac{1}{x^2} + 7}}$$

$$= \lim_{x \to -\infty} \frac{\frac{8}{x^2} + \frac{1}{x} - 4}{\sqrt{\frac{6}{x^4} + \frac{1}{x^2} + 7}}$$

As $x \to -\infty,\ \frac{8}{x^2} \to 0,\ \frac{1}{x} \to 0,\ \frac{6}{x^4} \to 0,\ \frac{1}{x^2} \to 0.$

$$= \lim_{x \to -\infty} \frac{0 + 0 - 4}{\sqrt{0 + 0 + 7}}$$

$$= \frac{-4}{\sqrt{7}}$$

$$\lim_{x \to -\infty} \frac{8 + x - 4x^2}{\sqrt{6 + x^2 + 7x^4}} = \frac{-4}{\sqrt{7}}$$

FINAL ANSWER

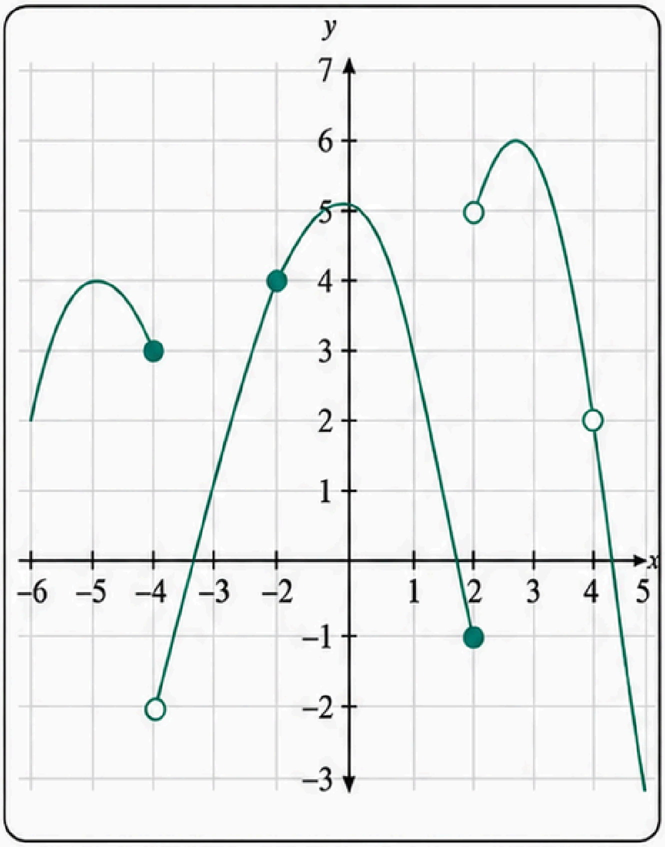
$$\lim_{x \to -\infty} \frac{8 + x - 4x^2}{\sqrt{6 + x^2 + 7x^4}} = \frac{-4}{\sqrt{7}}$$

CONTINUITY
Limits

DIFFICULTY: Medium | **Q #:** 2.12

QUESTION

Find all points of discontinuity.



SOLUTION

Check each critical number and endpoint.

- $x = -4$

$$\lim_{x \to -4^-} f(x) = 3$$

$$\lim_{x \to -4^+} f(x) = -2$$

$$f(-4) = 3$$

Limit does not exist.

Discontinuous at $x = -4$.

- $x = 2$

$$\lim_{x \to 2^-} f(x) = -1$$

$$\lim_{x \to 2^+} f(x) = 5$$

Since $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$, the limit does not exist.

Limit does not exist.

Ciscontinuous at $x = 2$.

- $x = 4$

$$\lim_{x \to 4} f(x) = 2$$

$f(4)$ is not defined

$f(4)$ is not defined.

Discontinuous at $x = 4$.

TIP

A function is discontinuous at a point a if at least one of the following is true:

1. $f(a)$ is not defined.
2. $\lim_{x \to a} f(x)$ does not exist.
3. $\lim_{x \to a} f(x) \neq f(a)$.

FINAL ANSWER

Therefore, the points of discontinuity are

$$x = -4,\ 2,\ 4.$$

1C1

SECTION 3

DERIVATIVES

Strengthen your foundation by mastering the core concepts of derivatives.

QUESTION

Using the definition of the derivative,

find the derivative of the function $f(x) = x^2$.

TIP

The derivative of a function $f(x)$ at a point x is defined by:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

SOLUTION

Using the definition of the derivative,

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{(x^2 + 2xh + h^2) - x^2}{h} \\
&= \lim_{h \to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h \to 0} \frac{h(2x+h)}{h} \\
&= \lim_{h \to 0} (2x + h) \\
&= 2x + \lim_{h \to 0} h \\
&= 2x + 0 \\
&= 2x
\end{aligned}$$

FINAL ANSWER

Therefore, $f'(x) = 2x$.

DIFFICULTY: Medium | Q #: **3.2**

QUESTION

Using the definition of the derivative,

find the derivative of the function $f(x) = 3x^3 - 2x + 1$.

SOLUTION

Using the definition of the derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left[3(x+h)^3 - 2(x+h) + 1\right] - \left(3x^3 - 2x + 1\right)}{h}$$

$$= \lim_{h \to 0} \frac{3(x^3 + 3x^2h + 3xh^2 + h^3) - 2x - 2h + 1 - 3x^3 + 2x - 1}{h}$$

$$= \lim_{h \to 0} \frac{3x^3 + 9x^2h + 9xh^2 + 3h^3 - 2x - 2h + 1 - 3x^3 + 2x - 1}{h}$$

$$= \lim_{h \to 0} \frac{9x^2h + 9xh^2 + 3h^3 - 2h}{h}$$

$$= \lim_{h \to 0} \frac{h(9x^2 + 9xh + 3h^2 - 2)}{h}$$

$$= \lim_{h \to 0} (9x^2 + 9xh + 3h^2 - 2)$$

$$= 9x^2 + 9x \lim_{h \to 0} h + 3 \lim_{h \to 0} h^2 - 2$$

$$= 9x^2 + 9x(0) + 3(0) - 2$$

$$= 9x^2 - 2$$

FINAL ANSWER

Therefore, $f'(x) = 9x^2 - 2$.

TIP

The derivative of a function $f(x)$ at a point x is defined by:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

DIFFICULTY: Hard | Q #: 3.3

QUESTION

Using the definition of the derivative, find the derivative of the function

$$f(x) = \frac{\sqrt{x}}{x-1}.$$

TIP

The derivative of a function $f(x)$ at a point x is defined by:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

SOLUTION

Using the definition of the derivative,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{\sqrt{x+h}}{x+h-1} - \frac{\sqrt{x}}{x-1}}{h} \\
&= \lim_{h\to 0} \frac{(x-1)\sqrt{x+h} - (x+h-1)\sqrt{x}}{h(x+h-1)(x-1)} \\
&= \lim_{h\to 0} \frac{(x-1)\sqrt{x+h} - (x-1+h)\sqrt{x}}{h(x+h-1)(x-1)} \\
&= \lim_{h\to 0} \frac{(x-1)\left(\sqrt{x+h} - \sqrt{x}\right) - h\sqrt{x}}{h(x+h-1)(x-1)} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h(x+h-1)} - \lim_{h\to 0} \frac{\sqrt{x}}{(x+h-1)(x-1)} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h(x+h-1)} - \frac{\sqrt{x}}{(x-1)^2} \\
&= \lim_{h\to 0} \frac{\left(\sqrt{x+h} - \sqrt{x}\right)\left(\sqrt{x+h} + \sqrt{x}\right)}{h(x+h-1)\left(\sqrt{x+h} + \sqrt{x}\right)} - \frac{\sqrt{x}}{(x-1)^2} \\
&= \lim_{h\to 0} \frac{x+h-x}{h(x+h-1)\left(\sqrt{x+h} + \sqrt{x}\right)} - \frac{\sqrt{x}}{(x-1)^2} \\
&= \lim_{h\to 0} \frac{h}{h(x+h-1)\left(\sqrt{x+h} + \sqrt{x}\right)} - \frac{\sqrt{x}}{(x-1)^2} \\
&= \lim_{h\to 0} \frac{1}{(x+h-1)\left(\sqrt{x+h} + \sqrt{x}\right)} - \frac{\sqrt{x}}{(x-1)^2} \\
&= \frac{1}{(x-1)(2\sqrt{x})} - \frac{\sqrt{x}}{(x-1)^2} \\
&= \frac{1}{2\sqrt{x}(x-1)} - \frac{\sqrt{x}}{(x-1)^2} \\
&= \frac{(x-1)-2x}{2\sqrt{x}(x-1)^2} \\
&= \frac{-x-1}{2\sqrt{x}(x-1)^2}
\end{aligned}$$

FINAL ANSWER

Therefore, $f'(x) = \dfrac{-x-1}{2\sqrt{x}(x-1)^2}$.

QUESTION

Find the derivative of each function.

(a) $f(x) = 3x^3 + 4x^2 - 5x + 2$

(b) $g(x) = 2x^2 - 7x + 3$

(c) $h(x) = x^3 - 4x^2 + x - 6$

TIP

Power Rule:

If $f(x) = x^n$, where n is a real number, then

$$f'(x) = \frac{d}{dx}(x^n) = nx^{n-1}.$$

SOLUTION

(a) $f(x) = 3x^3 + 4x^2 - 5x + 2$

$$\begin{aligned} f'(x) &= \frac{d}{dx}(3x^3 + 4x^2 - 5x + 2) \\ &= 3\frac{d}{dx}(x^3) + 4\frac{d}{dx}(x^2) - 5\frac{d}{dx}(x) + \frac{d}{dx}(2) \\ &= 3(3x^2) + 4(2x) - 5(1) + 0 \\ &= 9x^2 + 8x - 5 \end{aligned}$$

FINAL ANSWER: $f'(x) = 9x^2 + 8x - 5$

(b) $g(x) = 2x^2 - 7x + 3$

$$\begin{aligned} g'(x) &= \frac{d}{dx}(2x^2 - 7x + 3) \\ &= 2\frac{d}{dx}(x^2) - 7\frac{d}{dx}(x) + \frac{d}{dx}(3) \\ &= 2(2x) - 7(1) + 0 \\ &= 4x - 7 \end{aligned}$$

FINAL ANSWER: $g'(x) = 4x - 7$

(c) $h(x) = x^3 - 4x^2 + x - 6$

$$\begin{aligned} h'(x) &= \frac{d}{dx}(x^3 - 4x^2 + x - 6) \\ &= \frac{d}{dx}(x^3) - 4\frac{d}{dx}(x^2) + \frac{d}{dx}(x) - \frac{d}{dx}(6) \\ &= 3x^2 - 4(2x) + 1 - 0 \\ &= 3x^2 - 8x + 1 \end{aligned}$$

FINAL ANSWER: $h'(x) = 3x^2 - 8x + 1$

QUESTION

Find the derivative of each function.

(a) $f(x) = 3x^{\frac{1}{2}} - 2x^{-3} + \sqrt{x}$

(b) $h(x) = 7x^{\frac{5}{2}} - 4x^{-1} + 6x^{\frac{3}{4}}$

TIP

Power Rule:

If $f(x) = x^n$, where n is a real number, then

$$f'(x) = \frac{d}{dx}(x^n) = nx^{n-1}.$$

SOLUTION

(a) $f(x) = 3x^{\frac{1}{2}} - 2x^{-3} + \sqrt{x}$

$$= 3x^{\frac{1}{2}} - 2x^{-3} + x^{\frac{1}{2}}$$

$$f'(x) = \frac{d}{dx}\left(3x^{\frac{1}{2}}\right) + \frac{d}{dx}\left(-2x^{-3}\right) + \frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

$$= \frac{3}{2}x^{-\frac{1}{2}} + 6x^{-4} + \frac{1}{2}x^{-\frac{1}{2}}$$

$$= 2x^{-\frac{1}{2}} + 6x^{-4}$$

FINAL ANSWER: $f'(x) = 2x^{-\frac{1}{2}} + 6x^{-4}$

(b) $h(x) = 7x^{\frac{5}{2}} - 4x^{-1} + 6x^{\frac{3}{4}}$

$$h'(x) = \frac{d}{dx}\left(7x^{\frac{5}{2}}\right) + \frac{d}{dx}\left(-4x^{-1}\right) + \frac{d}{dx}\left(6x^{\frac{3}{4}}\right)$$

$$= 7\left(\frac{5}{2}x^{\frac{3}{2}}\right) + (-4)(-1)x^{-2} + 6\left(\frac{3}{4}x^{-\frac{1}{4}}\right)$$

$$= \frac{35}{2}x^{\frac{3}{2}} + 4x^{-2} + \frac{18}{4}x^{-\frac{1}{4}}$$

$$= \frac{35}{2}x^{\frac{3}{2}} + 4x^{-2} + \frac{9}{2}x^{-\frac{1}{4}}$$

FINAL ANSWER:

$$h'(c) = \frac{35}{2}x^{\frac{3}{2}} + 4x^{-2} + \frac{9}{2}x^{-\frac{1}{4}}$$

PRODUCT RULE
Derivatives

$\frac{d}{dx} f(x)$

DIFFICULTY: Easy | Q #: 3.6

QUESTION

Use the product rule to find the derivative of each function.

(a) $y = (2x^2 + 3x)(x - 4)$

(b) $y = (x^3 - 1)(x^2 + 2x)$

TIP

Product Rule:

If $y = f(x)g(x)$, then

$$\frac{d}{dx} f(x)g(x) = f(x)g'(x) + f'(x)g(x)$$

SOLUTION

(a) $y = (2x^2 + 3x)(x - 4)$

Let $f(x) = 2x^2 + 3x, \quad g(x) = x - 4$

$f'(x) = 4x + 3, \quad g'(x) = 1$

$$\begin{aligned}\frac{d}{dx} f(x)g(x) &= f(x)g'(x) + f'(x)g(x)\\ &= (2x^2 + 3x)(1) + (4x + 3)(x - 4)\\ &= 2x^2 + 3x + 4x^2 - 16x + 3x - 12\\ &= 6x^2 - 10x - 12\end{aligned}$$

FINAL ANSWER: $\frac{dy}{dx} = 6x^2 - 10x - 12$

(b) $y = (x^3 - 1)(x^2 + 2x)$

Let $f(x) = x^3 - 1, \quad g(x) = x^2 + 2x$

$f'(x) = 3x^2, \quad g'(x) = 2x + 2$

$$\begin{aligned}\frac{d}{dx} f(x)g(x) &= f(x)g'(x) + f'(x)g(x)\\ &= (x^3 - 1)(2x + 2) + (3x^2)(x^2 + 2x)\\ &= 2x^4 + 2x^3 - 2x - 2 + 3x^4 + 6x^3\\ &= 5x^4 + 8x^3 - 2x - 2\end{aligned}$$

FINAL ANSWER: $\frac{dy}{dx} = 5x^4 + 8x^3 - 2x - 2$

PRODUCT RULE
Derivatives

DIFFICULTY: Medium | Q #: **3.7**

QUESTION

Use the product rule to find the derivative of each function.

(a) $y = e^x \sin(x)$

(b) $y = \ln(x)\cos(x)$

TIP

Product Rule:

If $y = f(x)g(x)$, then

$$\frac{d}{dx} f(x)g(x) = f(x)g'(x) + f'(x)g(x)$$

SOLUTION

(a) $y = e^x \sin(x)$

Let $f(x) = e^x, \quad g(x) = \sin(x)$

$f'(x) = e^x, \quad g'(x) = \cos(x)$

$$\begin{aligned}\frac{d}{dx} f(x)g(x) &= f(x)g'(x) + f'(x)g(x) \\ &= e^x \cos(x) + e^x \sin(x) \\ &= e^x(\cos(x) + \sin(x))\end{aligned}$$

FINAL ANSWER: $\frac{dy}{dx} = e^x(\cos(x) + \sin(x))$

(b) $y = \ln(x)\cos(x)$

Let $f(x) = \ln(x), \quad g(x) = \cos(x)$

$f'(x) = \frac{1}{x}, \quad g'(x) = -\sin(x)$

$$\begin{aligned}\frac{d}{dx} f(x)g(x) &= f(x)g'(x) + f'(x)g(x) \\ &= \ln(x)(-\sin(x)) + \frac{1}{x}\cos(x) \\ &= -\ln(x)\sin(x) + \frac{\cos(x)}{x}\end{aligned}$$

FINAL ANSWER: $\frac{dy}{dx} = -\ln(x)\sin(x) + \frac{\cos(x)}{x}$

PRODUCT RULE
Derivatives

$\frac{d}{dx} f(x)$

DIFFICULTY: Hard | **Q #:** 3.8

QUESTION

Use the product rule to find the derivative of the function.

(a) $y = e^x \ln(x) \sin(x)$

TIP

Product Rule:

If $y = f(x)g(x)$, then

$$\frac{d}{dx} f(x)g(x) = f(x)g'(x) + f'(x)g(x)$$

SOLUTION

(a) $y = e^x \ln(x) \sin(x)$

Let $f(x) = e^x, \qquad g(x) = \ln(x)\sin(x)$

$$f'(x) = e^x, \qquad g'(x) = \frac{1}{x}\sin(x) + \ln(x)\cos(x)$$

$$\frac{d}{dx} f(x)g(x) = f(x)g'(x) + f'(x)g(x)$$

$$= e^x\left(\frac{1}{x}\sin(x) + \ln(x)\cos(x)\right) + e^x\ln(x)\sin(x)$$

$$= e^x\left(\frac{1}{x}\sin(x) + \ln(x)\cos(x) + \ln(x)\sin(x)\right)$$

$$= e^x\left(\frac{\sin(x)}{x} + \ln(x)\left(\sin(x) + \cos(x)\right)\right)$$

FINAL ANSWER:

$$\frac{dy}{dx} = e^x\left(\frac{\sin(x)}{x} + \ln(x)\left(\sin(x) + \cos(x)\right)\right)$$

QUOTIENT RULE
Derivatives

DIFFICULTY: Easy | Q #: 3.9

QUESTION

Use the quotient rule to find the derivative of each function.

(a) $y = \dfrac{2x+3}{x+1}$

(b) $y = \dfrac{x^2-1}{x}$

TIP

Quotient Rule:

If $y = \dfrac{f(x)}{g(x)}$, then

$$\frac{dy}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

SOLUTION

(a) $y = \dfrac{2x+3}{x+1}$

Let $f(x) = 2x+3, \quad g(x) = x+1$

$f'(x) = 2, \quad g'(x) = 1$

$$\begin{aligned}\frac{dy}{dx} &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \\ &= \frac{(2)(x+1) - (2x+3)(1)}{(x+1)^2} \\ &= \frac{2x+2-2x-3}{(x+1)^2} \\ &= \frac{-1}{(x+1)^2}\end{aligned}$$

FINAL ANSWER: $\dfrac{dy}{dx} = -\dfrac{1}{(x+1)^2}$

(b) $y = \dfrac{x^2-1}{x}$

Let $f(x) = x^2-1, \quad g(x) = x$

$f'(x) = 2x, \quad g'(x) = 1$

$$\begin{aligned}\frac{dy}{dx} &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \\ &= \frac{(2x)(x) - (x^2-1)(1)}{x^2} \\ &= \frac{2x^2 - (x^2-1)}{x^2} \\ &= \frac{x^2+1}{x^2}\end{aligned}$$

FINAL ANSWER: $\dfrac{dy}{dx} = \dfrac{x^2+1}{x^2}$

$\frac{d}{dx} f(x)$

QUOTIENT RULE
Derivatives

DIFFICULTY: Medium | **Q #:** 3.10

QUESTION

Use the quotient rule to find the derivative of each function.

(a) $y = \dfrac{\ln(x)}{\sin(x)}$

TIP

Quotient Rule:

If $y = \dfrac{f(x)}{g(x)}$, then

$$\frac{dy}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

SOLUTION

(a) $y = \dfrac{\ln(x)}{\sin(x)}$

Let $f(x) = \ln(x), \quad g(x) = \sin(x)$

$f'(x) = \dfrac{1}{x}, \quad g'(x) = \cos(x)$

$$\frac{dy}{dx} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$= \frac{\left(\frac{1}{x}\right)\sin(x) - \ln(x)\cos(x)}{[\sin(x)]^2}$$

$$= \frac{\frac{\sin(x)}{x} - \ln(x)\cos(x)}{\sin^2(x)}$$

$$= \frac{\sin(x) - x\ln(x)\cos(x)}{x\sin^2(x)}$$

FINAL ANSWER: $\dfrac{dy}{dx} = \dfrac{\sin(x) - x\ln(x)\cos(x)}{x\sin^2(x)}$

QUOTIENT RULE
Derivatives

$\frac{d}{dx} f(x)$

DIFFICULTY: Hard | Q #: **3.11**

QUESTION

Use the quotient rule to find the derivative of the function.

(a) $y = \dfrac{e^x \sin(x)}{x^2}$

TIP

Quotient Rule:

If $y = \dfrac{f(x)}{g(x)}$, then

$$\frac{dy}{dx} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

SOLUTION

(a) $y = \dfrac{e^x \sin(x)}{x^2}$

Let $f(x) = e^x \sin(x), \quad g(x) = x^2$

$$\begin{aligned} f'(x) &= (e^x \sin(x))' \\ &= (e^x)' \sin(x) + e^x (\sin(x))' \\ &= e^x \sin(x) + e^x \cos(x) \\ &= e^x (\sin(x) + \cos(x)) \end{aligned}$$

$$g'(x) = (x^2)' = 2x$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \\ &= \frac{e^x (\sin(x) + \cos(x))(x^2) - (e^x \sin(x))(2x)}{(x^2)^2} \\ &= \frac{x^2 e^x (\sin(x) + \cos(x)) - 2xe^x \sin(x)}{x^4} \\ &= \frac{e^x \left[x^2(\sin(x) + \cos(x)) - 2x \sin(x)\right]}{x^4} \end{aligned}$$

FINAL ANSWER:

$$\frac{dy}{dx} = \frac{e^x \left[x^2(\sin(x) + \cos(x)) - 2x \sin(x)\right]}{x^4}$$

QUESTION

Find the derivative of each function.

(a) $y = 5^x$

(b) $y = e^x$

(c) $y = \left(\frac{1}{3}\right)^x$

TIP

$$\frac{d}{dx}(a^x) = a^x \ln(a)$$

SOLUTION

(a) $y = 5^x$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(5^x)\\ &= 5^x \ln(5)\end{aligned}$$

(b) $y = e^x$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(e^x)\\ &= e^x \ln(e)\\ &= e^x \cdot 1\\ &= e^x\end{aligned}$$

(c) $y = \left(\frac{1}{3}\right)^x$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\left(\frac{1}{3}\right)^x\right)\\ &= \left(\frac{1}{3}\right)^x \ln\left(\frac{1}{3}\right)\end{aligned}$$

FINAL ANSWERS:

(a) $\frac{dy}{dx} = 5^x \ln(5)$ (b) $\frac{dy}{dx} = e^x$ (c) $\frac{dy}{dx} = \left(\frac{1}{3}\right)^x \ln\left(\frac{1}{3}\right)$

QUESTION

Use the chain rule to find the derivative of the function.

$$y = (3x+2)^5$$

TIP

Chain Rule:

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

SOLUTION

Let $f(u) = u^5$ and $g(x) = 3x + 2$

$$f'(u) = 5u^4$$

$$g'(x) = 3$$

By the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= f'(g(x))g'(x) \\ &= 5(3x+2)^4 \cdot 3 \\ &= 15(3x+2)^4\end{aligned}$$

FINAL ANSWER: $\frac{dy}{dx} = 15(3x+2)^4$

$\frac{d}{dx}f(x)$

CHAIN RULE
Derivatives

DIFFICULTY: Medium | **Q #:** 3.14

QUESTION

Use the chain rule to find the derivative of the function.

$$y = \sqrt[3]{1-8x}$$

TIP

Chain Rule:

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

SOLUTION

Let $f(u) = u^{1/3}$ and $g(x) = 1 - 8x$

$$f'(u) = \frac{1}{3}u^{-\frac{2}{3}}$$

$$g'(x) = -8$$

By the chain rule,

$$\begin{aligned} \frac{dy}{dx} &= f'(g(x))g'(x) \\ &= \frac{1}{3}(1-8x)^{-\frac{2}{3}} \cdot (-8) \\ &= -\frac{8}{3}(1-8x)^{-\frac{2}{3}} \\ &= -\frac{8}{3\sqrt[3]{(1-8x)^2}} \end{aligned}$$

FINAL ANSWER:

$$\frac{dy}{dx} = -\frac{8}{3\sqrt[3]{(1-8x)^2}}$$

CHAIN RULE Derivatives

DIFFICULTY: Hard

Q #: 3.15

QUESTION

Use the chain rule to find the derivative of the function.

$$y = \cos\left(x^2 e^x\right)$$

TIP

Chain Rule:

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

SOLUTION

Let $f(u) = \cos u$ and $g(x) = x^2 e^x$

$$f'(u) = -\sin u$$

$$g'(x) = \frac{d}{dx}\left(x^2 e^x\right) \qquad \text{[product rule]}$$

$$= 2x\, e^x + x^2 e^x$$

$$= e^x \left(2x + x^2\right)$$

By the chain rule,

$$\frac{dy}{dx} = f'(g(x))g'(x)$$

$$= -\sin\left(x^2 e^x\right) \cdot e^x \left(2x + x^2\right)$$

$$= -e^x \left(2x + x^2\right) \sin\left(x^2 e^x\right)$$

$$= -e^x\, x\,(x+2) \sin\left(x^2 e^x\right)$$

FINAL ANSWER: $$\frac{dy}{dx} = -e^x\, x\,(x+2) \sin\left(x^2 e^x\right)$$

CHAIN RULE Derivatives

DIFFICULTY: Hard

Q #: 3.16

QUESTION

Use the chain rule to find the derivative of the function.

$$y = \tan(\ln(x)\sin(x))$$

SOLUTION

Let $f(u) = \tan u$ and $g(x) = \ln(x)\sin(x)$

Reminder: $\frac{d}{du}\tan u = \sec^2 u$

$$f'(u) = \sec^2 u$$

$$g(x) = \ln(x)\sin(x)$$

$$g'(x) = \frac{d}{dx}(\ln(x)\sin(x)) \qquad \text{[product rule]}$$

$$= \left(\frac{1}{x}\right)\sin(x) + \ln(x)\cos(x)$$

$$= \frac{\sin(x)}{x} + \ln(x)\cos(x)$$

By the chain rule,

$$\frac{dy}{dx} = f'(g(x))g'(x)$$

$$= \sec^2(\ln(x)\sin(x)) \cdot \left(\frac{\sin(x)}{x} + \ln(x)\cos(x)\right)$$

FINAL ANSWER:

$$\frac{dy}{dx} = \sec^2(\ln(x)\sin(x)) \left(\frac{\sin(x)}{x} + \ln(x)\cos(x)\right)$$

TIP

Chain Rule:

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

QUESTION

Find $\frac{dy}{dx}$ by implicit differentiation.

a) $x^2 + y^2 = 25$

b) $xy + y = 1$

SOLUTION

a) Differentiate both sides with respect to x.

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

FINAL ANSWER: $\frac{dy}{dx} = -\frac{x}{y}$

b) Differentiate both sides with respect to x.

$$\frac{d}{dx}(xy) + \frac{d}{dx}(y) = \frac{d}{dx}(1)$$

$$x\frac{dy}{dx} + y + \frac{dy}{dx} = 0$$

$$(x+1)\frac{dy}{dx} + y = 0$$

$$(x+1)\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = -\frac{y}{x+1}$$

FINAL ANSWER: $\frac{dy}{dx} = -\frac{y}{x+1}$

DIFFICULTY: Hard

Q #: 3.18

QUESTION

Find $\frac{dy}{dx}$ by implicit differentiation.

$$x^2 + \ln(xy) = 2x$$

Then, find $\frac{dy}{dx}$ at the point $(1, e)$.

TIP

When differentiating implicit equations:

- Use the Chain Rule for terms containing y inside a function (e.g. $\ln(xy)$).
- Treat y as a function of x.
- Solve for $\frac{dy}{dx}$ before substituting the given point.

SOLUTION

Differentiate both sides with respect to x.

$$\frac{d}{dx}(x^2) + \frac{d}{dx}[\ln(xy)] = \frac{d}{dx}(2x)$$

$$2x + \frac{1}{xy}\frac{d}{dx}(xy) = 2$$

$$2x + \frac{1}{xy}\left(x\frac{dy}{dx} + y\right) = 2$$

- Multiply both sides by xy to clear the denominator.

$$2x^2y + x\frac{dy}{dx} + y = 2xy$$

- Solve for $\frac{dy}{dx}$.

$$x\frac{dy}{dx} = 2xy - 2x^2y - y$$

$$\frac{dy}{dx} = \frac{2xy - 2x^2y - y}{x}$$

$$\frac{dy}{dx} = 2y - 2xy - \frac{y}{x}$$

$$\boxed{\frac{dy}{dx} = 2y - 2xy - \frac{y}{x}}$$

- Evaluate at the point $(1, e)$.

$$\left.\frac{dy}{dx}\right|_{(1,e)} = 2(e) - 2(1)(e) - \frac{e}{1}$$

$$\left.\frac{dy}{dx}\right|_{(1,e)} = 2e - 2e - e$$

$$\left.\frac{dy}{dx}\right|_{(1,e)} = -e$$

FINAL ANSWER: $\left.\frac{dy}{dx}\right|_{(1,e)} = -e$

QUESTION

3.19. Find $\frac{dy}{dx}$ by logarithmic differentiation.

$$y = x^x$$

TIP

To use logarithmic differentiation:

- Take natural logs of both sides.
- Differentiate both sides implicitly with respect to x.
- Solve for $\frac{dy}{dx}$.

SOLUTION

Let $y = x^x$.

Take natural logs of both sides.

$$\ln y = \ln(x^x) = x \ln x$$

Differentiate both sides with respect to x.

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(x \ln x)$$

$$\frac{1}{y}\frac{dy}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{x}$$

$$\frac{1}{y}\frac{dy}{dx} = \ln x + 1$$

Solve for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = y(\ln x + 1)$$

Substitute $y = x^x$.

$$\boxed{\frac{dy}{dx} = x^x(\ln x + 1)}$$

FINAL ANSWER:

$$\boxed{\frac{dy}{dx} = x^x(\ln x + 1)}$$

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SECTION 4

DERIVATIVE APPLICATIONS

Apply derivative concepts to solve problems

CRITICAL POINTS
Derivative Applications

DIFFICULTY: Easy | Q #: **4.1**

QUESTION

4.1. Find the critical points of each function. Give your answers in exact form.

a) $f(x) = x^2 - 4x + 1$

b) $g(x) = x^3 - 3x$

TIP

Critical points occur where:

- $f'(x) = 0$ (stationary points)
- $f'(x)$ is undefined (e.g, cusps, vertical tangents)

Check both conditions.

SOLUTION

a) $f(x) = x^2 - 4x + 1$

$f'(x) = 2x - 4$

Set $f'(x) = 0$:

$2x - 4 = 0 \quad \Rightarrow \quad x = 2$

Point on curve: $f(2) = 2^2 - 4(2) + 1 = -3$

Final answer:
The critical point is $(2, -3)$.

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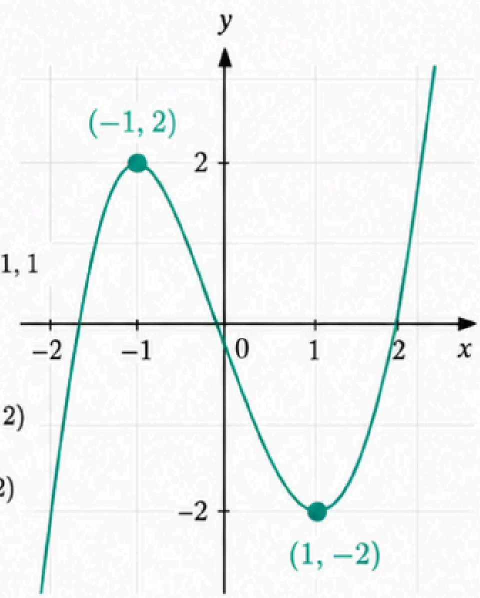
b) $g(x) = x^3 - 3x$

$g'(x) = 3x^2 - 3$



Set $g'(x) = 0$:

$3x^2 - 3 = 0 \;\Rightarrow\; x^2 = 1 \;\Rightarrow\; x = -1, 1$

Points on curve:

$g(-1) = (-1)^3 - 3(-1) = 2 \;\Rightarrow\; (-1, 2)$

$g(1) = (1)^3 - 3(1) = -2 \;\Rightarrow\; (1, -2)$

Final answer:
The critical points are $(-1, 2)$ and $(1, -2)$.

CRITICAL POINTS
Derivative Applications

DIFFICULTY: Medium | Q #: 4.2

QUESTION

4.2. Find the critical points of the function. Give your answers in exact form.

$$f(x) = \ln(x^2 + 4x + 14)$$

TIP

1. Find $f'(x)$ and set equal to 0.
2. Check where $f'(x)$ is undefined.
3. Evaluate $f(x)$ at critical x-values.

SOLUTION

$$f(x) = \ln(x^2 + 4x + 14)$$

$$f'(x) = \frac{1}{x^2 + 4x + 14} \cdot (2x + 4) = \frac{2x + 4}{x^2 + 4x + 14}$$

Set $f'(x) = 0$:

$$\frac{2x + 4}{x^2 + 4x + 14} = 0 \quad \Rightarrow \quad 2x + 4 = 0 \quad \Rightarrow \quad x = -2$$

Check for undefined points:

$$x^2 + 4x + 14 = (x + 2)^2 + 10 > 0 \quad \text{for all} \quad x \in \mathbb{R}.$$

So there are no values where $f'(x)$ is undefined.

Therefore, the only critical point occurs at $x = -2$.

Point on curve:

$$f(-2) = \ln((-2)^2 + 4(-2) + 14) = \ln(10)$$

Final answer: The critical point is $(-2, \ln 10)$.

QUESTION

4.3. Find the local maxima and local minima of each function. Give the coordinates of any extrema. Sketch the graph.

a) $f(x) = -x^2 + 4x + 1$

b) $g(x) = x^3 - 3x$

TIP

1. Find $f'(x)$ and set equal to 0.
2. Use $f''(x)$ to determine whether each critical point is a maximum or minimum.
3. Evaluate $f(x)$ at each critical point.

SOLUTION

a) $f(x) = -x^2 + 4x + 1$

$f'(x) = -2x + 4$

Set $f'(x) = 0$:

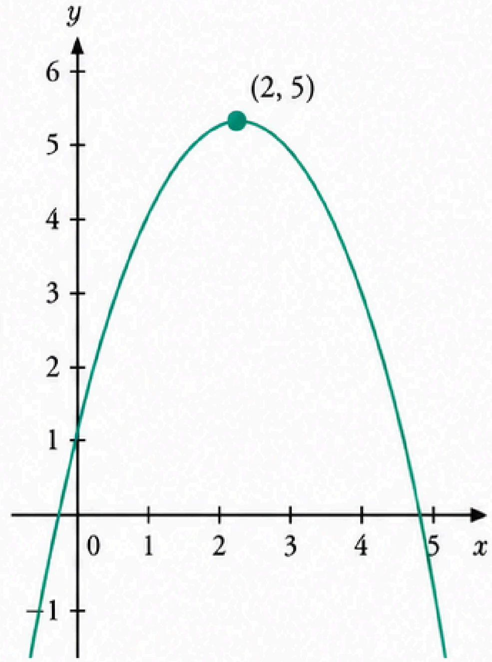
$-2x + 4 = 0 \quad \Rightarrow \quad x = 2$

$f''(x) = -2 < 0$, so f has a local maximum at $x = 2$.

$f(2) = -(2)^2 + 4(2) + 1 = 5$

Local maximum: $(2, 5)$

No local minimum.



b) $g(x) = x^3 - 3x$

$g'(x) = 3x^2 - 3$

Set $g'(x) = 0$:

$3x^2 - 3 = 0 \quad \Rightarrow \quad x^2 = 1 \quad \Rightarrow \quad x = -1, 1$

$g''(x) = 6x$

At $x = -1$: $g''(-1) = -6 < 0$, so g has a local maximum.

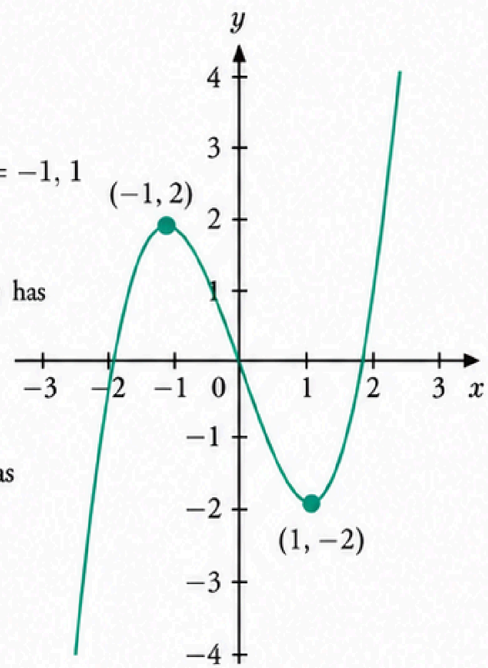
$g(-1) = (-1)^3 - 3(-1) = 2$

At $x = 1$: $g''(1) = 6 > 0$, so g has a local minimum.

$g(1) = (1)^3 - 3(1) = -2$

Local maximum: $(-1, 2)$

Local minimum: $(1, -2)$



QUESTION

4.4. Find the absolute maxima and absolute minima, as well as the local maxima and local minima, of the function on the closed interval $[-2, 3]$.

Give the coordinates of all extrema and sketch the graph.

$$f(x) = x^3 - 3x + 2$$

TIP

1. Find $f'(x)$ and set equal to 0.
2. Use $f''(x)$ to classify critical points as local maxima or minima.
3. Evaluate $f(x)$ at all critical points and at the endpoints of the interval.
4. Compare values to determine absolute (max/min) and local (max/min).

SOLUTION

Given $f(x) = x^3 - 3x + 2$ on $[-2, 3]$.

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x-1)(x+1)$$

Set $f'(x) = 0$:

$$3(x-1)(x+1) = 0 \quad \Rightarrow \quad x = -1, 1$$

$$f''(x) = 6x$$

At $x = -1$:

$f''(-1) = -6 < 0$

$\Rightarrow$ local maximum

At $x = 1$:

$f''(1) = 6 > 0$

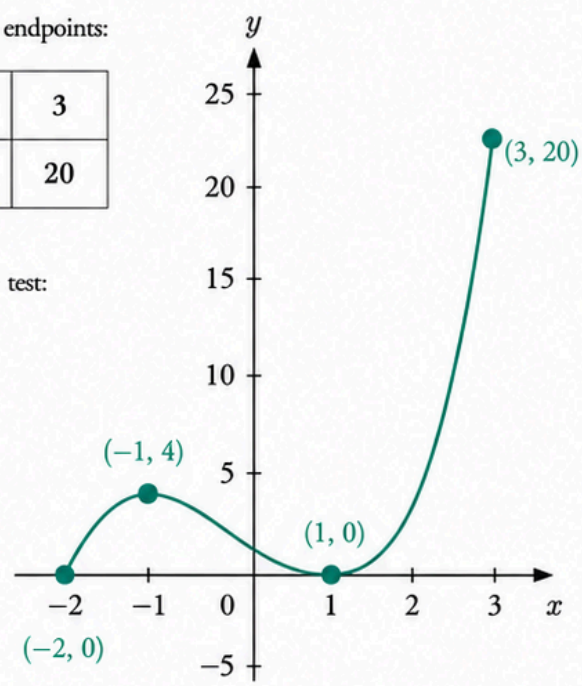
$\Rightarrow$ local minimum

Evaluate $f(x)$ at critical points and endpoints:

x	-2	-1	1	3
$f(x)$	0	4	0	20

From the table and second derivative test:

- Local maximum at $(-1, 4)$
- Local minimum at $(1, 0)$
- Absolute minimum at $(-2, 0)$
- Absolute maximum at $(3, 20)$



Absolute maximum: $(3, 20)$
Absolute minimum: $(-2, 0)$
Local maximum: $(-1, 4)$
Local minimum: $(1, 0)$

QUESTION

4.5. Let $f(x) = x^3 + 2x^2 - x$.

Verify that f satisfies the Mean Value Theorem on $[-1, 2]$.

Then find the value(s) of c in $(-1, 2)$ that are guaranteed by the theorem.

TIP

To apply the Mean Value Theorem:

1. Check that f is continuous on $[a, b]$ and differentiable on (a, b).
2. Compute the average rate of change:
$$\frac{f(b) - f(a)}{b - a}$$
3. Solve $f'(x) = \dfrac{f(b) - f(a)}{b - a}$ for c in (a, b).

SOLUTION

Let $f(x) = x^3 + 2x^2 - x$ and $[a, b] = [-1, 2]$.

1. Check the conditions.

$f(x)$ is a polynomial, so it is continuous on $[-1, 2]$ and differentiable on $(-1, 2)$.

Therefore, the Mean Value Theorem applies.

2. Compute the average rate of change on $[-1, 2]$.

$$f(-1) = (-1)^3 + 2(-1)^2 - (-1) = 2$$

$$f(2) = 2^3 + 2(2)^2 - 2 = 14$$

$$\frac{f(2) - f(-1)}{2 - (-1)} = \frac{14 - 2}{3} = \frac{12}{3} = 4$$

3. Find c such that $f'(c) = 4$.

$$f'(x) = 3x^2 + 4x - 1$$

Set $3x^2 + 4x - 1 = 4 \quad \Rightarrow \quad 3x^2 + 4x - 5 = 0$

Use the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ with $a = 3,\ b = 4,\ c = -5$:

$$x = \frac{-4 \pm \sqrt{4^2 - 4(3)(-5)}}{2(3)} = \frac{-4 \pm \sqrt{16 + 60}}{6} = \frac{-4 \pm \sqrt{76}}{6}$$

$$= \frac{-4 \pm 2\sqrt{19}}{6} = \frac{-2 \pm \sqrt{19}}{3}$$

Since $c \in (-1, 2)$, the only valid value is $c = \dfrac{-2 + \sqrt{19}}{3}$

(the other root $\dfrac{-2 - \sqrt{19}}{3} \approx -2.12$ is not in $(-1, 2)$).

Answer: The value guaranteed by the Mean Value Theorem is $c = \dfrac{-2 + \sqrt{19}}{3}$.

QUESTION

4.6. Evaluate each limit using L'Höpital's Rule.

1. $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$

2. $\lim\limits_{x \to 1} \dfrac{x^2 - 1}{x - 1}$

3. $\lim\limits_{x \to \infty} \dfrac{\ln x}{x}$

TIP

L'Höpital's Rule applies when:

- $\lim\limits_{x \to a} f(x) = 0$ and $\lim\limits_{x \to a} g(x) = 0$, or
- $\lim\limits_{x \to a} |f(x)| = \infty$ and $\lim\limits_{x \to a} |g(x)| = \infty$.

If those forms occur, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is $\pm\infty$).

SOLUTION

1. $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$

As $x \to 0$, $\sin x \to 0$ and $x \to 0$, so the form is $0/0$.

Apply L'Höpital's Rule:

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = \cos 0 = 1$$

Answer: 1

2. $\lim\limits_{x \to 1} \dfrac{x^2 - 1}{x - 1}$

As $x \to 1$, $x^2 - 1 \to 0$ and $x - 1 \to 0$, so the form is $0/0$.

Apply L'Höpital's Rule:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{2x}{1} = 2(1) = 2$$

Answer: 2

3. $\lim\limits_{x \to \infty} \dfrac{\ln x}{x}$

As $x \to \infty$, $\ln x \to \infty$ and $x \to \infty$, so the form is ∞/∞.

Apply L'Höpital's Rule:

$$\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1/x}{1} = \lim_{x \to \infty} \frac{1}{x} = 0$$

Answer: 0

QUESTION

4.7. Evaluate each limit using L'Höpital's Rule.

1. $\displaystyle\lim_{x\to 0} \frac{\ln(1+x)-x}{x^2}$

2. $\displaystyle\lim_{x\to\infty} \frac{2x}{\sqrt{x+1}}$

SOLUTION

1. $\displaystyle\lim_{x\to 0} \frac{\ln(1+x)-x}{x^2}$

As $x \to 0$, $\ln(1+x) - x \to 0$ and $x^2 \to 0$, so the form is $0/0$.

Apply L'Höpital's Rule.

$$\lim_{x\to 0} \frac{\ln(1+x)-x}{x^2} = \lim_{x\to 0} \frac{\frac{1}{1+x}-1}{2x}$$

As $x \to 0$, $\frac{1}{1+x} - 1 \to 0$ and $2x \to 0$, so the form is $0/0$.

Apply L'Höpital's Rule again.

$$\lim_{x\to 0} \frac{\frac{1}{1+x}-1}{2x} = \lim_{x\to 0} \frac{-\frac{1}{(1+x)^2}}{2} = -\frac{1}{2}$$

Answer: $-\frac{1}{2}$

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2. $\displaystyle\lim_{x\to\infty} \frac{2x}{\sqrt{x+1}}$

As $x \to \infty$, $2x \to \infty$ and $\sqrt{x+1} \to \infty$, so the form is ∞/∞.

Apply L'Höpital's Rule.

$$\lim_{x\to\infty} \frac{2x}{\sqrt{x+1}} = \lim_{x\to\infty} \frac{2}{\frac{1}{2\sqrt{x+1}}} = \lim_{x\to\infty} 4\sqrt{x+1}$$

As $x \to \infty$, $4\sqrt{x+1} \to \infty$ and $1 \to 1$, so the form is $\infty/1$.

Therefore, $\displaystyle\lim_{x\to\infty} \frac{2x}{\sqrt{x+1}} = \frac{\infty}{1} = \infty$

Answer: ∞

1C1

SECTION 5

INTEGRALS AND APPLICATIONS

Build a foundation on integrals
and learn to solve problems

QUESTION

5.1. Evaluate each indefinite integral.

1. $\int x^2 \, dx$

2. $\int \cos x \, dx$

3. $\int e^x \, dx$

TIP

Integrals are the opposite of differentiation.

If $\frac{d}{dx}[F(x)] = f(x)$, then

$$\int f(x) \, dx = F(x) + C,$$

where C is an arbitrary constant.

SOLUTION

1. $\int x^2 \, dx$

Using the power rule in reverse:

$$\int x^2 \, dx = \frac{x^3}{3} + C$$

Answer: $\frac{x^3}{3} + C$

2. $\int \cos x \, dx$

Using the basic integral:

$$\int \cos x \, dx = \sin x + C$$

Answer: $\sin x + C$

3. $\int e^x \, dx$

Using the basic integral:

$$\int e^x \, dx = e^x + C$$

Answer: $e^x + C$

QUESTION

5.2. Evaluate each indefinite integral.

1. $\int 3x^2 \, dx$

2. $\int (5x + 4) \, dx$

3. $\int (x^3 - 2x + 1) \, dx$

TIP

Integral Power Rule

For any $n \neq -1$,

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

This rule is used to integrate polynomial terms.

SOLUTION

1. $\int 3x^2 \, dx$

Using the power rule for integration:

$$\int 3x^2 \, dx = 3 \int x^2 \, dx = 3 \cdot \frac{x^3}{3} + C = x^3 + C$$

Answer: $x^3 + C$

2. $\int (5x + 4) \, dx$

Integrate term by term:

$$\int (5x + 4) \, dx = \int 5x \, dx + \int 4 \, dx$$

$$= \frac{5x^2}{2} + 4x + C$$

Answer: $\frac{5x^2}{2} + 4x + C$

3. $\int (x^3 - 2x + 1) \, dx$

Integrate term by term:

$$\int (x^3 - 2x + 1) \, dx = \int x^3 \, dx - \int 2x \, dx + \int 1 \, dx$$

$$= \frac{x^4}{4} - x^2 + x + C$$

Answer: $\frac{x^4}{4} - x^2 + x + C$

QUESTION

5.3. Evaluate each indefinite integral.

1. $\int \left(\sqrt{x} + 2x - \frac{1}{\sqrt{x}}\right) dx$

2. $\int \left(x^{-2} + 3x^{-1/2} - 4\right) dx$

3. $\int (\sin x + 3\cos x - x)\, dx$

SOLUTION

1. $\int \left(\sqrt{x} + 2x - \frac{1}{\sqrt{x}}\right) dx$

Rewrite using exponents: $\int \left(x^{1/2} + 2x - x^{-1/2}\right) dx$

Integrate term by term:

$$\int x^{1/2}\, dx + \int 2x\, dx - \int x^{-1/2}\, dx$$
$$= \frac{2}{3}x^{3/2} + x^2 - 2x^{1/2} + C$$

Answer: $\frac{2}{3}x^{3/2} + x^2 - 2x^{1/2} + C$

2. $\int \left(x^{-2} + 3x^{-1/2} - 4\right) dx$

Integrate term by term:

$$\int x^{-2}\, dx + \int 3x^{-1/2}\, dx - \int 4\, dx$$
$$= -x^{-1} + 3 \cdot 2x^{1/2} - 4x + C$$
$$= -\frac{1}{x} + 6\sqrt{x} - 4x + C$$

Answer: $-\frac{1}{x} + 6\sqrt{x} - 4x + C$

3. $\int (\sin x + 3\cos x - x)\, dx$

Integrate term by term:

$$\int \sin x\, dx + \int 3\cos x\, dx - \int x\, dx$$
$$= -\cos x + 3\sin x - \frac{1}{2}x^2 + C$$

Answer: $-\cos x + 3\sin x - \frac{1}{2}x^2 + C$

QUESTION

5.4. Evaluate each definite integral.

1. $\displaystyle\int_0^2 (3x+1)\,dx$

2. $\displaystyle\int_{-1}^1 x^2\,dx$

3. $\displaystyle\int_0^{\pi/2} \cos x\,dx$

TIP

To evaluate a definite integral:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is any antiderivative of $f(x)$.

SOLUTION

1. $\displaystyle\int_0^2 (3x+1)\,dx$

Antiderivative: $F(x) = \frac{3}{2}x^2 + x$

$$F(2) - F(0) = \left(\frac{3}{2}(2)^2 + 2\right) - \left(\frac{3}{2}(0)^2 + 0\right)$$

$$= (6+2) - (0) = 8$$

Answer: 8

2. $\displaystyle\int_{-1}^1 x^2\,dx$

Antiderivative: $F(x) = \frac{x^3}{3}$

$$F(1) - F(-1) = \left(\frac{1^3}{3}\right) - \left(\frac{(-1)^3}{3}\right)$$

$$= \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}$$

Answer: $\frac{2}{3}$

3. $\displaystyle\int_0^{\pi/2} \cos x\,dx$

Antiderivative: $F(x) = \sin x$

$$F\left(\frac{\pi}{2}\right) - F(0) = \sin\left(\frac{\pi}{2}\right) - \sin(0)$$

$$= 1 - 0 = 1$$

Answer: 1

QUESTION

5.5. Evaluate each definite integral.

1. $\int_0^1 (e^x - x^3)\, dx$

2. $\int_0^{\pi/2} (\sin x + \cos x)\, dx$

3. $\int_1^2 \left(\frac{1}{x} - 3x^2\right) dx$

SOLUTION

1. $\int_0^1 (e^x - x^3)\, dx$

Antiderivative: $F(x) = e^x - \frac{x^4}{4}$

Evaluate: $F(1) - F(0) = \left(e - \frac{1}{4}\right) - (1 - 0)$

$$= e - \frac{5}{4}$$

Answer: $e - \frac{5}{4}$

2. $\int_0^{\pi/2} (\sin x + \cos x)\, dx$

Antiderivative: $F(x) = -\cos x + \sin x$

Evaluate: $F\left(\frac{\pi}{2}\right) - F(0) = \left(-\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) - (-\cos 0 + \sin 0)$

$$= (0 + 1) - (-1 + 0) = 2$$

Answer: 2

3. $\int_1^2 \left(\frac{1}{x} - 3x^2\right) dx$

Antiderivative: $F(x) = \ln|x| - x^3$

Evaluate: $F(2) - F(1) = (\ln 2 - 8) - (\ln 1 - 1)$

$$= (\ln 2 - 8) - (0 - 1) = \ln 2 - 7$$

Answer: $\ln 2 - 7$

QUESTION

5.6. Evaluate each integral using *u* substitution.

1. $\int 2x(x^2+1)^3\,dx$

2. $\int 3(3x-2)^2\,dx$

3. $\int 5x(x^2+4)\,dx$

TIP

Steps for *u* substitution:

1. Choose *u* as the inner function.
2. Find *du* by differentiating *u*.
3. Solve for *du* and substitute into the integral.
4. Rewrite the integral in terms of *u*.
5. Integrate with respect to *u*.
6. Substitute back for *u*.
7. Add + C.

SOLUTION

1. $\int 2x(x^2+1)^3\,dx$

Let $u = x^2+1$

$du = 2x\,dx$

$$\int 2x(x^2+1)^3\,dx = \int u^3\,du = \frac{u^4}{4} + C$$

Substitute back: $\frac{(x^2+1)^4}{4} + C$

Answer: $\frac{(x^2+1)^4}{4} + C$

2. $\int 3(3x-2)^2\,dx$

Let $u = 3x-2$

$du = 3\,dx \Rightarrow dx = \frac{du}{3}$

$$\int 3(3x-2)^2\,dx = \int 3u^2 \cdot \frac{du}{3} = \int u^2\,du = \frac{u^3}{3} + C$$

Substitute back: $\frac{(3x-2)^3}{3} + C$

Answer: $\frac{(3x-2)^3}{3} + C$

3. $\int 5x(x^2+4)\,dx$

Let $u = x^2+4$

$du = 2x\,dx \Rightarrow x\,dx = \frac{du}{2}$

$$\int 5x(x^2+4)\,dx = \int 5 \cdot \frac{du}{2} = \frac{5}{2}\int u\,du = \frac{5}{2} \cdot \frac{u^2}{2} + C$$

Substitute back: $\frac{5}{4}(x^2+4)^2 + C$

Answer: $\frac{5}{4}(x^2+4)^2 + C$

QUESTION

5.7. Evaluate each integral using u substitution.

1. $\int \frac{\ln x}{x}\, dx$

2. $\int \sin x \cos x \, dx$

3. $\int \tan x \sec^2 x \, dx$

TIP

For u substitution:

1. Choose u as the inner function.
2. Find du by differentiating u.
3. Solve for du and substitute into the integral.
4. Rewrite the integral in terms of u.
5. Integrate with respect to u.
6. Substitute back for u.
7. Add + C.

SOLUTION

1. $\int \frac{\ln x}{x}\, dx$

Let $u = \ln x$

$du = \frac{1}{x}\, dx \Rightarrow dx = x\, du$

$$\int \frac{\ln x}{x}\, dx = \int \frac{u}{x}(x\, du) = \int u\, du = \frac{u^2}{2} + C$$

Substitute back: $\frac{(\ln x)^2}{2} + C$

Answer: $\frac{(\ln x)^2}{2} + C$

2. $\int \sin x \cos x \, dx$

Let $u = \sin x$

$du = \cos x \, dx$

$$\int \sin x \cos x \, dx = \int u\, du = \frac{u^2}{2} + C$$

Substitute back: $\frac{\sin^2 x}{2} + C$

Answer: $\frac{\sin^2 x}{2} + C$

3. $\int \tan x \sec^2 x \, dx$

Let $u = \tan x$

$du = \sec^2 x \, dx$

$$\int \tan x \sec^2 x \, dx = \int u\, du = \frac{u^2}{2} + C$$

Substitute back: $\frac{\tan^2 x}{2} + C$

Answer: $\frac{\tan^2 x}{2} + C$

QUESTION

5.8. Evaluate the integral using u substitution.

$$\int 3t^{-4}\left(2+4t^{-3}\right)^{-7}\,dt$$

SOLUTION

Evaluate $\int 3t^{-4}\left(2+4t^{-3}\right)^{-7}\,dt$ using u substitution.

Step 1: Let $u = 2 + 4t^{-3}$

$$du = -12t^{-4}\,dt \quad\Rightarrow\quad t^{-4}\,dt = -\frac{1}{12}\,du$$

Step 2: Substitute into the integral.

$$\begin{aligned}&\int 3t^{-4}\left(2+4t^{-3}\right)^{-7}\,dt\\ &= \int 3u^{-7}\left(t^{-4}\,dt\right)\\ &= \int 3u^{-7}\left(-\frac{1}{12}\,du\right)\\ &= -\frac{1}{4}\int u^{-7}\,du\end{aligned}$$

Step 3: Integrate with respect to u.

$$\begin{aligned}-\frac{1}{4}\int u^{-7}\,du &= -\frac{1}{4}\cdot\frac{u^{-6}}{-6}+C\\ &= \frac{1}{24}u^{-6}+C\end{aligned}$$

Step 4: Substitute back for u.

$$\frac{1}{24}u^{-6}+C = \frac{1}{24}\left(2+4t^{-3}\right)^{-6}+C$$

Answer: $\frac{1}{24}\left(2+4t^{-3}\right)^{-6}+C$

QUESTION

5.9. Evaluate the integrals using integration by parts.

(a) $\int xe^x \, dx$

(b) $\int_1^2 \ln(x) \, dx$

TIP

Integration by Parts Formula:

$$\int u \, dv = uv - \int v \, du$$

SOLUTION

(a) Let $u = x \qquad dv = e^x \, dx$

$du = dx \qquad v = e^x$

$$\int xe^x \, dx = xe^x - \int e^x \, dx$$

$$= xe^x - e^x + C$$

Answer: $xe^x - e^x + C$

(b) Let $u = \ln(x) \qquad dv = dx$

$du = \frac{1}{x} \, dx \qquad v = x$

$$\int \ln(x) \, dx = \left[x \ln(x) - x\right]$$

$$\int_1^2 \ln(x) \, dx = \left[x \ln(x) - x\right]_1^2$$

$$= (2 \ln 2 - 2) - (1 \ln 1 - 1)$$

$$= (2 \ln 2 - 2) - (0 - 1)$$

$$= 2 \ln 2 - 2 + 1$$

$$= 2 \ln 2 - 1$$

Answer: $2 \ln 2 - 1$

QUESTION

5.10. Evaluate the integrals using integration by parts.

(a) $\int x\, e^{2x}\, dx$

(b) $\int x^2\, e^x\, dx$

SOLUTION

(a) Let $u = x \qquad dv = e^{2x}\, dx$

$du = dx \qquad v = \frac{1}{2} e^{2x}$

$$\int x\, e^{2x}\, dx = x\left(\frac{1}{2} e^{2x}\right) - \int \frac{1}{2} e^{2x}\, dx$$

$$= \frac{x}{2} e^{2x} - \frac{1}{2}\left(\frac{1}{2} e^{2x}\right) + C$$

$$= \frac{x}{2} e^{2x} - \frac{1}{4} e^{2x} + C$$

Answer: $\frac{e^{2x}}{4}(2x - 1) + C$

(b) Let $u = x^2 \qquad dv = e^x\, dx$

$du = 2x\, dx \qquad v = e^x$

$$\int x^2\, e^x\, dx = x^2\, e^x - \int 2x\, e^x\, dx$$

Let $I = \int 2x\, e^x\, dx$

Let $u = 2x \qquad dv = e^x\, dx$

$du = 2\, dx \qquad v = e^x$

$$I = 2x\, e^x - \int 2\, e^x\, dx$$

$$= 2x\, e^x - 2\, e^x + C$$

$$\int x^2\, e^x\, dx = x^2\, e^x - \left(2x\, e^x - 2\, e^x\right) + C$$

$$= x^2\, e^x - 2x\, e^x + 2\, e^x + C$$

$$= e^x\left(x^2 - 2x + 2\right) + C$$

Answer: $e^x\left(x^2 - 2x + 2\right) + C$

QUESTION

5.11. Evaluate the integral using integration by parts.

$$\int_0^{\frac{\pi}{4}} x \sin 2x \ dx$$

SOLUTION

Let $u = x \qquad dv = \sin 2x \ dx$

$du = dx \qquad v = -\frac{1}{2}\cos 2x$

$$\int_0^{\frac{\pi}{4}} x \sin 2x \, dx = \left[-\frac{1}{2} x \cos 2x\right]_0^{\pi/4} - \int_0^{\frac{\pi}{4}} \left(-\frac{1}{2}\cos 2x\right) dx$$

$$= \left[-\frac{1}{2} x \cos 2x\right]_0^{\pi/4} + \frac{1}{2}\int_0^{\frac{\pi}{4}} \cos 2x \ dx$$

$$= \left[-\frac{1}{2} x \cos 2x\right]_0^{\pi/4} + \frac{1}{2}\left[\frac{1}{2}\sin 2x\right]_0^{\pi/4}$$

$$= \left[-\frac{1}{2} x \cos 2x\right]_0^{\pi/4} + \frac{1}{4}\left[\sin 2x\right]_0^{\pi/4}$$

Evaluate the bounds:

$$\left[-\frac{1}{2} x \cos 2x\right]_0^{\pi/4} = -\frac{1}{2}\left(\frac{\pi}{4}\cos\frac{\pi}{2}\right) - \left(-\frac{1}{2}(0)\cos 0\right)$$

$$= 0 - 0 = 0$$

$$\frac{1}{4}\left[\sin 2x\right]_0^{\pi/4} = \frac{1}{4}\left(\sin\frac{\pi}{2} - \sin 0\right) = \frac{1}{4}(1-0) = \frac{1}{4}$$

Hence,

$$\int_0^{\frac{\pi}{4}} x \sin 2x \ dx = 0 + \frac{1}{4} = \frac{1}{4}$$

Answer: $\frac{1}{4}$

QUESTION

5.12. Find the area of the region bounded by $y = \sqrt{x}$ and $y = x^2$ from $x = 0$ to $x = 1$.

TIP

If $f(x) \geq g(x)$ on $[a, b]$, then the area between $y = f(x)$ (top curve) and $y = g(x)$ (bottom curve) from $x = a$ to $x = b$ is

$$\int_a^b [f(x) - g(x)]\, dx.$$

SOLUTION

Step 1: Find the points of intersection.

$$\sqrt{x} = x^2 \;\Rightarrow\; x^{1/2} = x^2 \;\Rightarrow\; x^4 = x$$

$$x(x^3 - 1) = 0 \;\Rightarrow\; x = 0,\ 1$$

On $[0,\ 1]$, $\sqrt{x} \geq x^2$.

Step 2: Set up the integral for the area.

$$A = \int_0^1 \left(\sqrt{x} - x^2\right)\, dx$$

Step 3: Evaluate the integral.

$$A = \int_0^1 \left(\sqrt{x} - x^2\right)\, dx$$

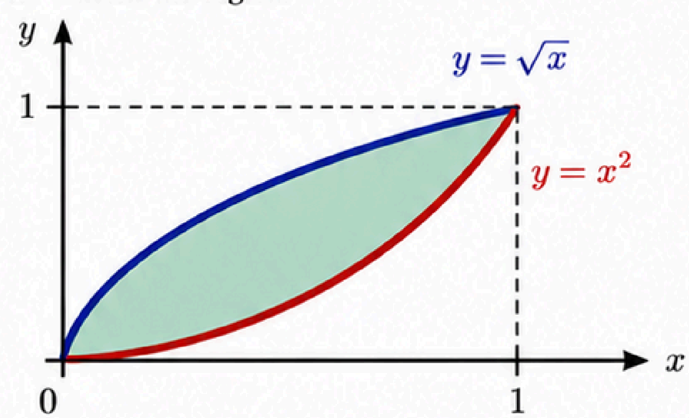
$$= \int_0^1 \left(x^{1/2} - x^2\right)\, dx$$

$$= \left[\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right]_0^1$$

$$= \left(\frac{2}{3} - \frac{1}{3}\right) - (0 - 0)$$

$$= \frac{1}{3}$$

Step 4: Sketch the region.



The shaded region is the area between $y = \sqrt{x}$ (top) and $y = x^2$ (bottom) from $x = 0$ to $x = 1$.

Answer: $\frac{1}{3}$

QUESTION

5.13. Find the area of the region bounded by $x = \frac{1}{2}y^2 - 3$ and $y = x - 1$ from $y = -2$ to $y = 4$.

TIP

When the region is described with respect to y (i.e., from $y = c$ to $y = d$), write x in terms of y for each curve.

If $x_{\text{right}}(y) \geq x_{\text{left}}(y)$ on $[c, d]$, then the area is

$$\int_c^d \left[x_{\text{right}}(y) - x_{\text{left}}(y)\right]\, dy.$$

SOLUTION

Step 1: Set up the integral for the area.

The line: $y = x - 1 \quad \Rightarrow \quad x = y + 1$

The parabola: $x = \frac{1}{2}y^2 - 3$ (already in terms of y)

On $[-2, 4]$, the line is to the right of the parabola.

Thus,

$$\begin{aligned} A &= \int_{-2}^{4} \left[x_{\text{right}}(y) - x_{\text{left}}(y)\right]\, dy \\ &= \int_{-2}^{4} \left[(y + 1) - \left(\frac{1}{2}y^2 - 3\right)\right] dy \\ &= \int_{-2}^{4} \left(-\frac{1}{2}y^2 + y + 4\right) dy \end{aligned}$$

Step 2: Evaluate the integral.

$$\begin{aligned} A &= \left[-\frac{1}{6}y^3 + \frac{1}{2}y^2 + 4y\right]_{-2}^{4} \\ &= \left(-\frac{1}{6}(4)^3 + \frac{1}{2}(4)^2 + 4(4)\right) - \left(-\frac{1}{6}(-2)^3 + \frac{1}{2}(-2)^2 + 4(-2)\right) \\ &= \left(-\frac{64}{6} + 8 + 16\right) - \left(\frac{8}{6} + 2 - 8\right) \\ &= \frac{16}{3} - \left(-\frac{10}{3}\right) = \frac{26}{3} = 18 \end{aligned}$$

Answer: 18

THANK YOU

FOR USING THIS CALCULUS 1 SURVIVAL PACK!

I hope this resource helped you learn, practice,
and build the confidence you need to succeed in Calculus 1.

Keep up the hard work!